5 Coordinate systems

5.1 Introduction

Coordinate systems provide the mathematical underpinnings for spatial operations in multidimensional space. A coordinate system assigns coordinates to points in space and/or time. This International Standard defines coordinate systems within the scope of three conceptual spaces: coordinate-space, position-space, and object-space. ***Coordinate-spaces*** specify the sets of coordinate n-tuples that form the domains of coordinate systems. ***Position-spaces*** are abstract Euclidean vector spaces that provide the mathematical and geometric foundation needed to define spatial operations. ***Object-spaces*** are Euclidean vector spaces associated with specific spatial objects of interest, such as the Earth, a building, or a vehicle. Coordinate-spaces, position-spaces, and object-spaces, and the relationships among them, are normatively defined in 5.2.

An ***abstract coordinate system*** specifies a function, termed a ***generating function***, which assigns unique n-tuples in a domain in coordinate-space to points in an m-dimensional position-space $(1 \leq n \leq m \leq 3)$. Abstract coordinate systems are normatively defined, and many types of abstract coordinate systems are specified, in 5.3.

A ***spatial coordinate system*** extends the assignment of unique coordinate n-tuples from points in a position-space to points in an object-space. The assignment function combines an abstract coordinate system generating function with a ***normal embedding*** that maps the orthonormal frame within position-space to a corresponding orthonormal frame within object-space. Spatial coordinate systems are normatively defined in 5.4. The relationships among coordinate-space, position-space, abstract coordinate systems, object-space, and spatial coordinate systems are shown in Figure 5.23.

The ability of a spatial coordinate system to assign a unique coordinate to a point in an object-space assumes that the position of the point in object-space is static. In a dynamic system, that assumption may not hold unless the spatial coordinate system is associated with a particular moment in time. ***Temporal coordinate systems*** provide a standard way of associating time with a spatial coordinate system. Temporal coordinate systems are normatively defined in 5.5.

5.2 Coordinate-space, position-space, and object-space

5.2.1 Coordinate-space

A *coordinate*[5] is an ordered n-tuple $(1 \leq n \leq 3)$. A *coordinate-component*[6] is an individual element of a coordinate n-tuple. The k^{th} *coordinate-component* $(1 \leq k \leq n)$ is the k^{th} component of a coordinate n-tuple. A coordinate system may optionally specify coordinate-component names and symbols in a specified order. In 3D coordinate systems, the 3rd coordinate-component may be identified as the *vertical coordinate-component*.

A *coordinate-space* specifies a set of coordinate n-tuples that forms the domain of a coordinate system. Such coordinate n-tuples include Cartesian (x, y, z), polar (ρ, θ), cylindrical (ρ, θ, h), and geodetic (λ, φ, h). A coordinate-space may include constraints on coordinate n-tuple components in such domains.

[5] The ISO 19111 term for this concept is “coordinate tuple”.

[6] The ISO 19111 term for this concept is “coordinate”.

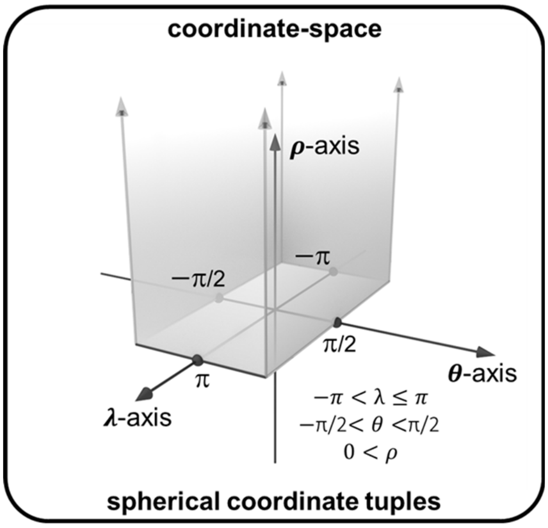


Figure 5.1 — A coordinate-space (including the domain for spherical coordinate n-tuples)

Figure 5.1 illustrates the structure of a coordinate-space for 3D spherical coordinate tuples of the form (λ, θ, ρ). The coordinate-components of these tuples are:

λ: longitude in radians, such that $-\pi < \lambda \leq \pi$,

θ: spherical latitude in radians, such that $-{}^{\pi}/_{2} < \theta < {}^{\pi}/_{2}$, and

ρ: radius in metres, such that $0 < \rho$.

This coordinate-space defines the domain for a spherical coordinate system (see 5.3.8.4) as a subset (highlighted in grey) of the coordinate-space

5.2.2 Position-space

Position-space of dimension m, $(1 \leq n \leq m \leq 3)$, is the Euclidean vector space $\mathbb{R}^m$ as defined in A.2. Mathematical concepts of $\mathbb{R}^m$ as a vector space, the point-set topology of $\mathbb{R}^m$, the theory of real-valued functions on $\mathbb{R}^m$, and algebraic and analytic geometry, including the concepts of point, line, and plane, are all assumed and hold.

Position-space serves as a mathematical abstraction of object-spaces so that the methods of linear algebra and multivariate calculus can be applied to spatial concepts, including abstract coordinate systems and the computational aspects of spatial operations. The purpose of position-space is to provide flexibility in applying different types of coordinate systems to object-spaces for many different types of spatial objects of interest.

The *position* of a point is the displacement of that point with respect to a designated reference point, called the origin. Each point in Euclidean vector space is associated with the position vector that extends from the origin to that point with length equal to the Euclidean distance between the origin and that point. Thus, points in Euclidean space and position vectors with respect to the origin are equivalent concepts. The position of an object is typically expressed in terms of the position of a representative point within the object.

A *direction* in a Euclidean vector space is represented by a unit vector. *A vector quantity*, expressing a physical measurement such as velocity or acceleration (at a given instant in time), is represented by a direction vector combined with a magnitude. Velocity is a vector quantity that expresses the rate of change of position. Acceleration is a vector quantity that expresses the rate of change of velocity.

An ordered set of m mutually perpendicular unit vectors forms a canonical Cartesian basis for position-space. This Cartesian basis allows positions, directions, vector quantities, and distance measurements in position-space to be quantified.

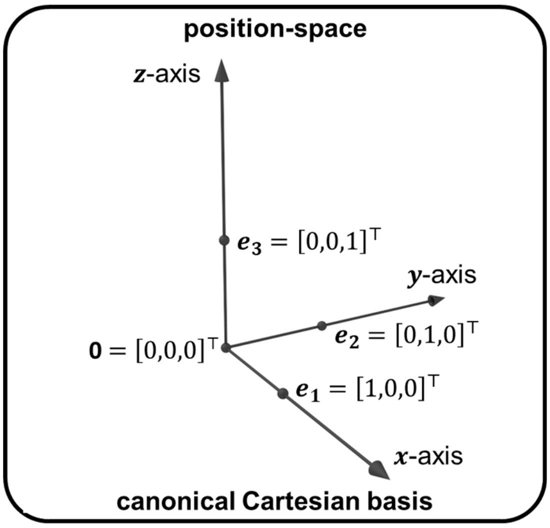


Figure 5.2 — 3D position-space and its canonical Cartesian basis

Figure 5.2 illustrates 3D position-space, showing its origin, and the unit vectors that form its canonical Cartesian basis.

Position vectors in 2 and 3 dimensions are denoted as $[x,y]^\top \equiv \begin{bmatrix} x \\ y \end{bmatrix}$ and $[x,y,z]^\top \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, respectively (see A.2).

These components, unless otherwise indicated, are specified with respect to the canonical Cartesian basis and origin.

The canonical origin for $\mathbb{R}^2$ is the zero vector $\mathbf{0} = [0,0]^\top$. The canonical Cartesian basis vectors for $\mathbb{R}^2$ are $\boldsymbol{e}_1 = [1,0]^\top, \boldsymbol{e}_2 = [0,1]^\top$.

The canonical origin for $\mathbb{R}^3$ is the zero vector $\mathbf{0} = [0,0,0]^\top$. The canonical Cartesian basis vectors for $\mathbb{R}^3$ are $\boldsymbol{e}_1 = [1,0,0]^\top$, $\boldsymbol{e}_2 = [0,1,0]^\top$, $\boldsymbol{e}_3 = [0,0,1]^\top$.

5.2.3 Orthonormal frames

An *orthonormal frame* within a Euclidean vector space, in 2 or 3 dimensions, consists of an origin vector $\boldsymbol{q}$ and an ordered set of mutually perpendicular unit basis vectors $\boldsymbol{r}, \boldsymbol{s}$, and, in the 3D case, $\boldsymbol{t}$. These vectors form the basis for a Cartesian coordinate system. Each of the vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ (and $\boldsymbol{t}$) is specified in the Euclidean vector space. Any vector $\boldsymbol{p}$ with respect to the Euclidean vector space origin corresponds to the vector $\tilde{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{q}$ with respect to the orthonormal frame origin $\boldsymbol{q}$. In the 3D case, $\tilde{\boldsymbol{p}} = u\boldsymbol{r} + v\boldsymbol{s} + w\boldsymbol{t}$ in terms of the orthonormal frame basis vectors. The 3-tuple $[u,v,w]$ is termed the coordinate of $\tilde{\boldsymbol{p}}$. In the 2D case, $\tilde{\boldsymbol{p}} = u\boldsymbol{r} + v\boldsymbol{s}$ and the coordinate is the tuple $[u,v]$. In terms of the Euclidian vector space, $\boldsymbol{p} = \boldsymbol{q} + u\boldsymbol{r} + v\boldsymbol{s} + w\boldsymbol{t}$ in the 3D case, and $\boldsymbol{p} = \boldsymbol{q} + u\boldsymbol{r} + v\boldsymbol{s}$ in the 2D case. The 3D case is depicted in Figure 5.3.

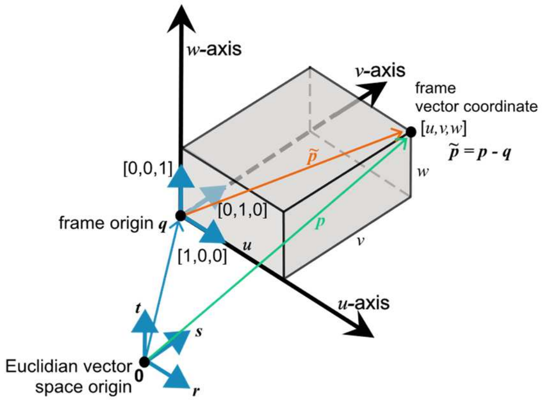


Figure 5.3 — A right-handed orthonormal frame

EXAMPLE The canonical Cartesian basis for 3D position-space is an orthonormal frame specified with $\boldsymbol{q} = \mathbf{0}$, $\boldsymbol{r} = \boldsymbol{e}_1, \boldsymbol{s} = \boldsymbol{e}_2, \boldsymbol{t} = \boldsymbol{e}_3$.

A 3D orthonormal frame is termed right-handed if the vertices of the triangle formed by its basis unit vectors are in clockwise order when viewed from the origin, as defined in ISO 80000-2, and shown in Figure 5.3. In this International Standard, all 3D orthonormal frames shall be right-handed.

5.2.4 Object-space

Object-space is the Euclidean vector space (a universe[7]) that is fixed to a designated spatial object of interest. Object-space provides the application domain context for spatial concepts including positions, directions, vector quantities, and orientations.

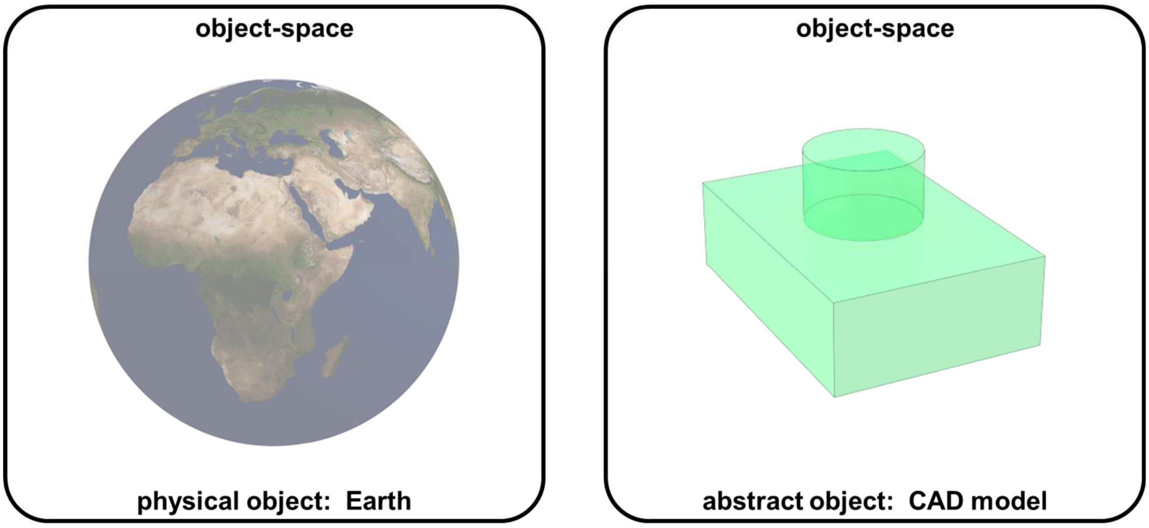


Figure 5.4 — Object-spaces for the Earth and for a CAD model

The spatial objects of concern in this International Standard include physical and abstract objects, as illustrated in Figure 5.4. *Physical objects* are real-world objects, such as Earth or a building. The length of one

[7] The set of all continuations of a spatial object is termed the universe of the object. In physics, this is termed “the space of the object”. [EINS]

metre has intrinsic meaning in the object-space of a physical object. *Abstract objects* are conceptual objects including engineering, mathematical, and virtual models. A length of one metre does not have intrinsic meaning in the object-spaces of abstract objects. Thus, to relate abstract object-spaces to other (physical or abstract) object-spaces, each abstract object-space is required to have a designated length scale.

At any given instance in time, the position of a point in object-space is fixed with respect to the spatial object of interest. This is done either by a time-invariant constant or a time-dependent function. If points and the spatial object of interest have a time-dependent relationship, the positions of the points shall be qualified by a time value. Thus, at a specified time, the points and the spatial object of interest have a fixed spatial relationship.

EXAMPLE 1 The Sun and the Earth are both physical objects. In the object-space of the Sun, the Sun is the spatial object of interest and is fixed and the Earth moves according to a time dependent function. In the object-space of the Earth, the Earth is the spatial object of interest and is fixed and the Sun moves according to a different time dependent function.

EXAMPLE 2 At any given time the International Space Station (ISS) has a unique and unambiguous position in the object-space of the Earth.

EXAMPLE 3 At any given time each component of the ISS has a fixed position in the object-space of the ISS.

EXAMPLE 4 A solar collector component of the ISS was manufactured in compliance with an engineering model. The engineering model was designed in the object-space of an abstract CAD/CAM model. The physical solar collector was constructed in its own physical object-space.

An object-space is a Euclidian space. In general, however, an object-space is not a vector space. Once a point in object-space is designated as an origin point, it becomes a vector space with respect to that origin and all points in the object-space are vectors, each with length and direction given as the distance and direction of the point from the origin.

5.2.5 Normal embeddings

A *normal embedding* is a distance-preserving function mapping vectors in position-space to points in an object-space of the same dimension. A function $\boldsymbol{E}$ from position-space to object-space is *distance-preserving* if for any two positions $\boldsymbol{p}$ and $\boldsymbol{q}$ in position-space, the Euclidean distance $d(\boldsymbol{p}, \boldsymbol{q})$ is equal to the measured distance in object-space from $\boldsymbol{E}(\boldsymbol{p})$ to $\boldsymbol{E}(\boldsymbol{q})$ in metres. The distance-preserving property implies that a normal embedding is one-to-one and continuous. Normal embeddings also preserve angles and areas.

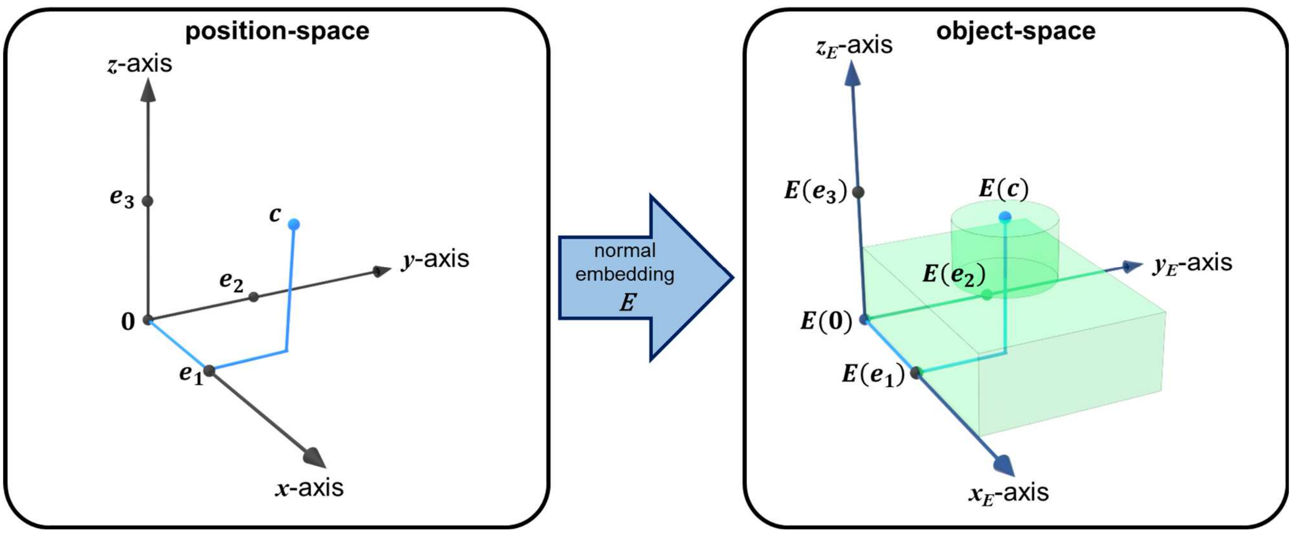


Figure 5.5 — A normal embedding that maps position-space to an object-space

Position-space together with a normal embedding provides a specific algebraic model of an object-space by determining an orthonormal frame within the object space. This frame is termed the *embedded frame* and is determined as follows. In the 3-dimentional case, as shown in Figure 5.5, the position-space orthonormal frame is formed by the origin $\mathbf{0}$ and unit basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. The normal embedding $\boldsymbol{E}$ forms an orthonormal frame within object-space with origin $\boldsymbol{E}(\mathbf{0})$ and basis vectors $\boldsymbol{E}(\boldsymbol{e}_1)$, $\boldsymbol{E}(\boldsymbol{e}_2)$, and $\boldsymbol{E}(\boldsymbol{e}_3)$. Since $\boldsymbol{E}$ is distance preserving, these vectors are orthogonal unit vectors, thus an embedded frame is an orthonormal frame and $\boldsymbol{E}$ is then an isomorphism between position-space and object-space. A normal embedding of a 3D position-space is *right-handed* if this frame is a right-handed frame. Normal embeddings for 2-dimensional object-space form orthonormal frames in a similar way.

The point $\boldsymbol{E}(\mathbf{0})$ is termed the *origin of the normal embedding* $\boldsymbol{E}$. The point $\boldsymbol{E}(\boldsymbol{e}_1)$ is the x_E*-axis unit point* of the normal embedding $\boldsymbol{E}$. Depending on the dimension of position-space, $\boldsymbol{E}(\boldsymbol{e}_2)$ is the y_E*-axis unit point* and $\boldsymbol{E}(\boldsymbol{e}_3)$ is the z_E*-axis unit point*. Normal embeddings are used to relate abstract coordinate systems for position-space to spatial coordinate systems for an object-space (see 5.4).

There are infinitely many normal embeddings of an n-dimensional position-space for a given object-space, depending on placement of the origin and direction of the axes.

There are infinitely many ways to select the origin of the embedding in the object-space. The origin can be located at any point within the spatial object of interest, at any point on its surface, or at any point nearby in space. Common selections include the centre of mass of the object, its geometric centre, or a corner of the object (assuming it has corners) or its bounding volume such that the object is completely within the first octant.

Given a selected origin, there are infinitely many ways to orient the axes. If the object is a celestial body, the axes could be aligned with its rotational axis, its magnetic field axis, or the direction of the closest star (such as the Sun). If the object is a vehicle, the axes could be aligned based on its direction of forward motion or other common reference orientations. If the object is located on, or near, the surface of the Earth, common selections include east-north-up (ENU) and north-east-down (NED).

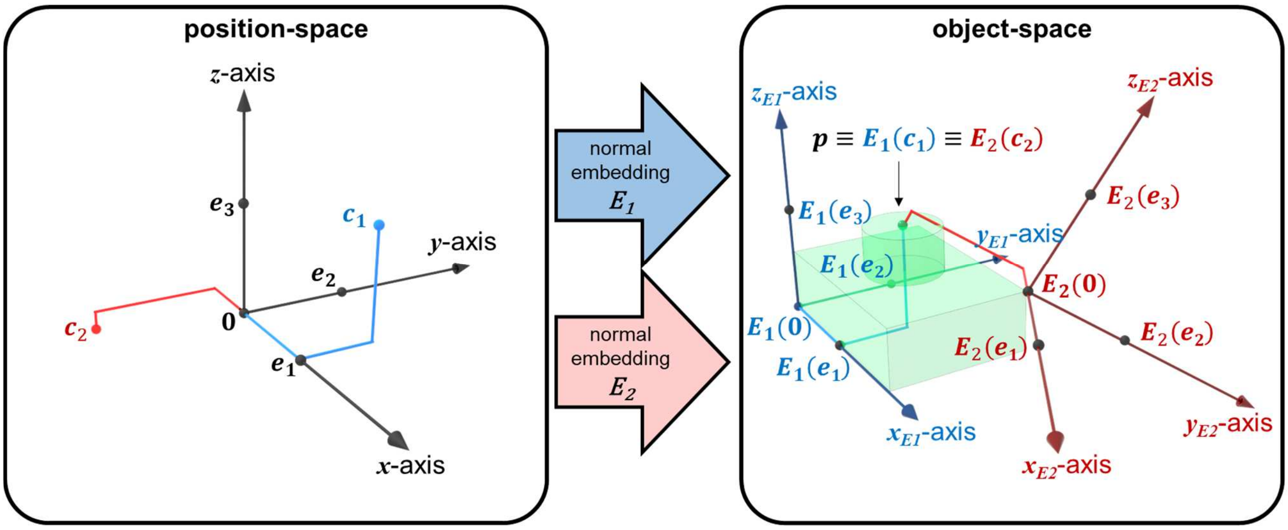


Figure 5.6 — Two distinct normal embeddings that map position-space to an object-space

Figure 5.6 illustrates two distinct normal embeddings for a given object-space, each determining a different embedded frame. Each embedding assigns the origin ($\mathbf{0}$) to different points on the spatial object of interest ($\boldsymbol{E}_1(\mathbf{0})$ and $\boldsymbol{E}_2(\mathbf{0})$, respectively), and assigning the basis vectors ($\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$) to different directions ($\boldsymbol{E}_1(\boldsymbol{e1}), \boldsymbol{E}_1(\boldsymbol{e2}), \boldsymbol{E}_1(\boldsymbol{e3})$ and $\boldsymbol{E}_2(\boldsymbol{e1}), \boldsymbol{E}_2(\boldsymbol{e2}), \boldsymbol{E}_2(\boldsymbol{e3})$, respectively)) relative to the object, providing two distinct algebraic models of that object-space. In the figure, the two embedded frames are depicted in two different colours. In the object space, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ refer to the same point $\boldsymbol{p} \equiv \boldsymbol{E}_1(\boldsymbol{c}_1)) \equiv \boldsymbol{E}_2(\boldsymbol{c}_2)$ on the object, expressed in each of the two embedded frames.

A similarity transformation is used to express the relationship between one embedded frame with respect to a second embedded frame within the same object-space A similarity transformation consists of a translation, a rotation, and/or a scaling operation. If $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ are two normal embeddings, there exists a similarity transformation $\boldsymbol{H_{E2\leftarrow E1}}$ such that $\boldsymbol{E_2}$ is the composition of $\boldsymbol{E_1}$ with $\boldsymbol{H_{E2\leftarrow E1}}$, i.e., $\boldsymbol{E}_2 = \boldsymbol{H_{E2\leftarrow E1}} \circ \boldsymbol{E}_1$. This is depicted in Figure 5.6, where a point $\boldsymbol{p}$ in object-space will have vector coordinates $[x_1, y_1, z_1]_{\boldsymbol{E1}}$ and $[x_2, y_2, z_2]_{\boldsymbol{E2}}$ in the $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ embedded frames respectively. The similarity transformation $\boldsymbol{H_{E2\leftarrow E1}}$, operating on object-space, that will translate $\boldsymbol{E_1}(\mathbf{0})$ to $\boldsymbol{E_2}(\mathbf{0})$ and align the $\boldsymbol{E_1}$ basis axes with $\boldsymbol{E_2}$ basis axes will also perform a change of basis operation: $[x_2, y_2, z_2]_{\boldsymbol{E2}} = H([x_1, y_1, z_1]_{\boldsymbol{E1}})$. Thus, similarity transformations can be used in the transformation of coordinates between orthonormal frames. Similarity transformations are addressed in greater detail in 7.3.2.

The method of specifying a normal embedding varies across disciplines and application domains. In some application domains, a normal embedding is implicitly defined by the specification of the origin point and axis directions. In the case of geodesy, an origin point at the centre of the Earth cannot be directly specified. Instead, its location is implied by specifying other geometric entities from physical measurements. An object reference model (see 7.4) implicitly identifies a unique normal embedding in this manner. Other disciplines use a variety of techniques to either implicitly or explicitly define a normal embedding. This International Standard encapsulates these techniques within the concepts of reference datum and object reference model.

5.3 Abstract coordinate systems

5.3.1 Introduction

An abstract coordinate system assigns a unique coordinate n-tuple to each point in a range of position vectors in an m-dimensional Euclidean vector space $(1 \le n \le m \le 3)$ termed position-space, which has a canonical basis that defines an orthonormal frame. The assignment function is termed the generating function of the abstract coordinate system. The range may encompass the entire vector space or a proper sub-set, such as a surface or a curve.

Abstract coordinate systems are formally defined in 5.3.2. Abstract coordinate systems are characterized by type (5.3.3) and properties (5.3.5). In addition, abstract coordinate systems for 3D position-space generate coordinate-component surfaces (5.3.4). Localization operators modify abstract coordinate systems by shifting the vector space origin and changing axis directions (5.3.6). Map projections and augmented map projections are treated as a special case of abstract coordinate systems and have additional classifications and properties, as well as several functions unique to map projections (5.3.7). The elements for the specification of an abstract coordinate system, along with standardized abstract coordinate systems, are specified in 5.3.8. In this International Standard the term “coordinate system (CS)”, if not otherwise qualified, is defined to mean “abstract CS.”

This International Standard takes a functional approach to the construction of coordinate systems. Annex A provides a concise summary of mathematical concepts and specifies the notational conventions used in this International Standard. In particular, Annex A defines the terms interior, one-to-one, smooth, smooth surface, smooth curve, orientation-preserving, and connected. Additionally, a newly introduced concept, replete, will be used. A set D is replete if all points in D belong to the closure of the interior of D (see Annex A). A replete set is a generalization of an open set that allows the inclusion of boundary points. Boundary points are important in the definitions of certain coordinate systems.

5.3.2 Definition

An *abstract coordinate system* (CS) assigns a unique coordinate to each point in a subset of position-space (5.2.2). An abstract Coordinate System shall be comprised of:

a) a CS domain in n-dimensional coordinate-space, $(1 \le n \le 3)$,

b) a generating function, and

c) a CS range in m-dimensional position-space, $(n \le m \le 3)$, where:

a) The *CS domain* shall be a connected replete domain in n-dimensional coordinate-space, the space of n-tuples. The elements of the CS domain are coordinates.

b) The *generating function* assigns each coordinate to a point in position-space. It shall be a one-to-one, smooth function (see Annex A.4) from the CS domain onto the generating function range.

c) The generating function range shall be termed the *CS range*. When $n = 2$ and $m = 3$, the CS range shall be a subset of a smooth surface[8]. When $n = 1$ and $m = 2$ or 3, the CS range shall be a subset of an implicitly specified smooth curve[9]. The elements of the CS range are positions.

The *coordinate of a position* $\boldsymbol{p}$ shall be the unique coordinate whose generating function value is $\boldsymbol{p}$.

The generating function may be parameterized. The generating function parameters (if any) shall be termed the *CS parameters.*

The inverse of the generating function shall be termed the *inverse generating function*. The inverse generating function is one-to-one and is smooth in the interior of the CS domain, except at points in the image of the CS domain boundary points where it may be discontinuous.

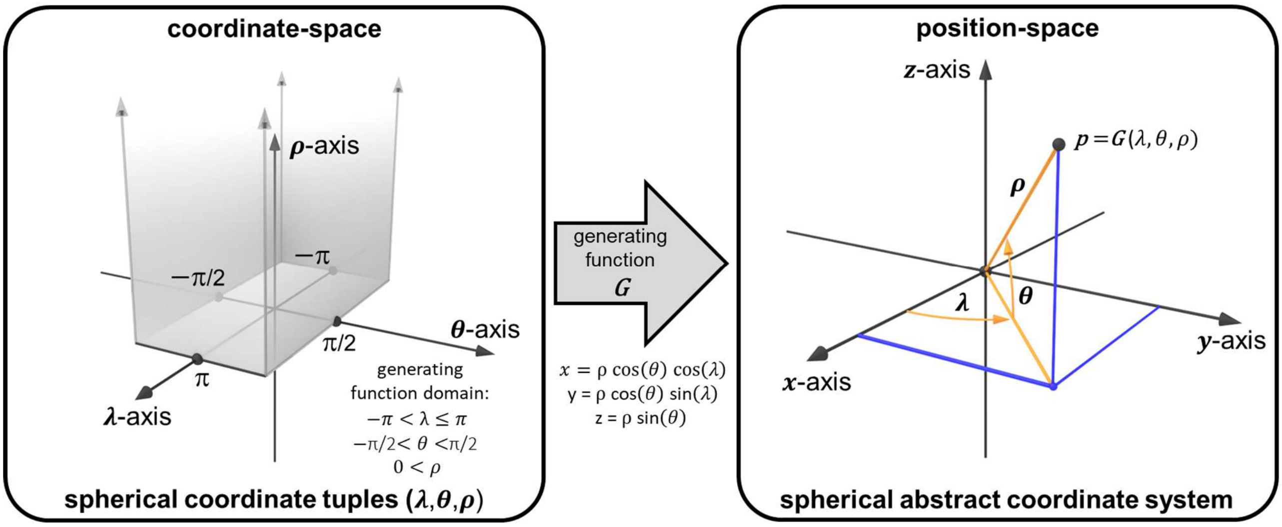


Figure 5.7 — Abstract equatorial spherical coordinate system example

Figure 5.7 illustrates the components of the Equatorial Spherical abstract coordinate system.

NOTE 1 The generating function of a CS is often specified by an algebraic and/or trigonometric description of a geometric relationship (see 5.3.3 Example 2). There are also CSs that do not have geometric derivations. The Mercator map projection (see Table 5.18) is specified to satisfy a functional requirement of conformality (see 5.3.7.3.2) rather than by a geometric construction.

[8] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth surface. This requirement specifies that there exists one smooth surface for all of the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

[9] The generating function properties and the implicit function theorem together imply that for each point in the interior of the CS domain, there is an open neighbourhood of the point whose image under the generating function lies in a smooth curve. This requirement specifies that there exists one implicitly-defined smooth curve for all the points in the CS domain. This requirement is specified to exclude mathematically pathological cases.

5.3.3 Coordinate system types

The coordinate-space and position-space dimensions characterize an abstract CS by *CS type* as defined in Table 5.1.

Table 5.1 — Coordinate system types

CS type	Dimension of coordinate-space	Dimension of position-space
3D	3	3
surface	2	3
curve[10]	1	3
2D	2	2
plane curve[10]	1	2
1D	1	1

For brevity, a CS may be referred to by its CS type as a 3D CS, surface CS, curve CS, 2D CS, plane curve CS, or 1D CS.

EXAMPLE 1 The identity function on 3D Euclidean space is the generating function of the Euclidean 3D coordinate system. Both the coordinate system domain and range sets are the entire Euclidean space. In this case, coordinate-space and position-space are one and the same. The Euclidean 3D coordinate system (see 5.3.8.2) is a linear coordinate system.

EXAMPLE 2 The polar coordinate system is an example of an abstract coordinate system of coordinate system type 2D that is defined with the generating function:

$$\boldsymbol{G}\big((\rho,\ \theta)\big) = [x, y]^{\mathsf{T}}$$

where:

$$x = \rho\cos(\theta)\,,\ \ y = \rho\sin(\theta).$$

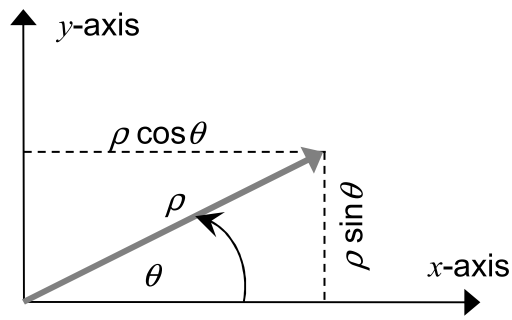


Figure 5.8 — Polar CS geometry

The geometric and trigonometric relationships for this generating function are illustrated in Figure 5.8
The CS domain in coordinate-space is $\{(\rho,\ \theta)\ in\ \mathbb{R}^2 | 0 < \rho, 0 \le \theta < 2\pi\} \cup \{(0,0)\}$.
The CS range in position-space is $\mathbb{R}^2$.

[10] The ISO 19111 concept of a linear coordinate system, defined as “a one-dimensional coordinate system in which a linear feature forms the axis”, is similar in some respects to the curve CS and plane curve CS concepts. This ISO 19111 concept is distinct from the linearity property of abstract coordinate systems (see 5.3.5.1).

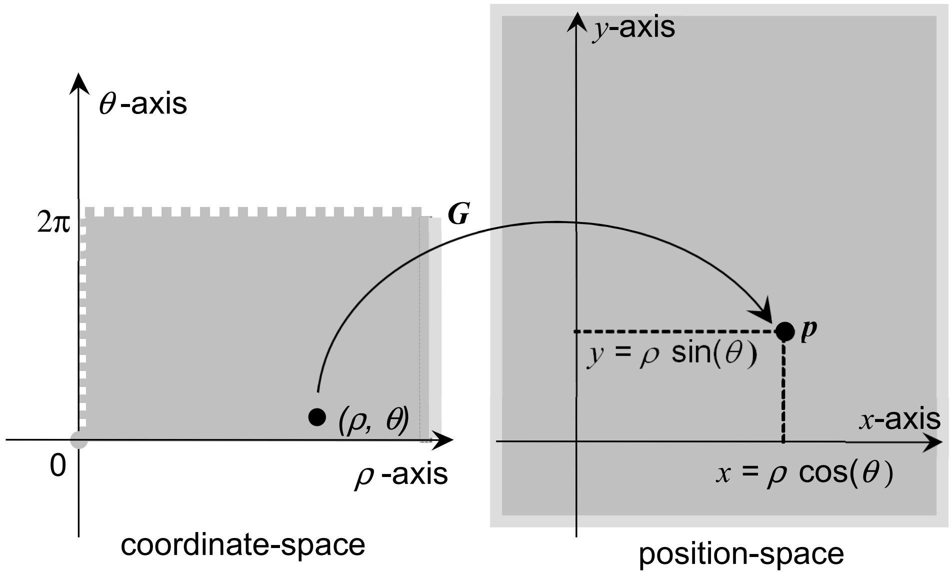


Figure 5.9 — The Polar CS generating function

This generating function, $\boldsymbol{G}$, is illustrated in Figure 5.9. The grey boxes with lighter grey edges in this figure represent the fact that the range in position-space extends indefinitely, and that the domain in coordinate-space extends indefinitely in the direction of the ρ-axis. The dashed grey edges indicate an open boundary, and a solid edge is a closed boundary. This CS range, CS domain, and generating function define an abstract CS representing polar coordinates as defined in mathematics [EDM, "Coordinates"]. This coordinate system is fully specified in 5.3.8.27 as the Polar coordinate system.

NOTE 2 In the special case where 1) the CS domain and CS range are both $\mathbb{R}^n$ and 2) the function is the identity function, this approach to defining coordinate systems reduces to the usual definition of the Euclidean coordinate system on $\mathbb{R}^n$ where each point is identified by an n-tuple of real numbers [EDM] (see Table 5.8, Table 5.29 and Table 5.35).

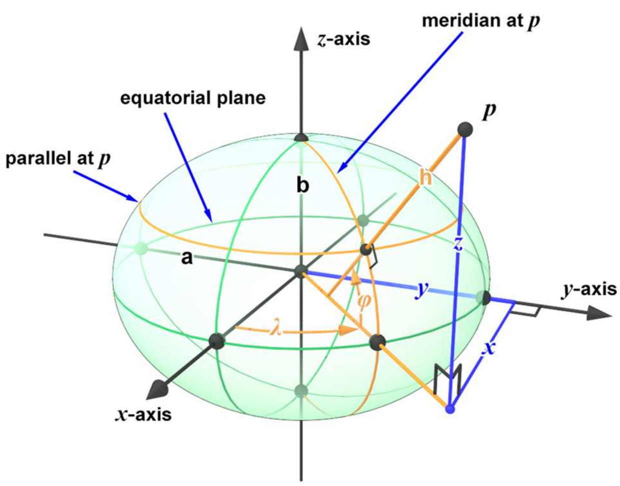


Figure 5.10 — The geodetic coordinate system geometric and trigonometric relationships

EXAMPLE 3 The geodetic coordinate system for positions in the space containing an oblate ellipsoid is an example of a coordinate system of coordinate system type 3D. The geometric and trigonometric relationships for the generating function of this coordinate system are illustrated in Figure 5.10. The coordinate system parameters are the major and minor semi-axis values a and b. This coordinate system is fully specified in 5.3.8.8 as the Geodetic 3D coordinate system.

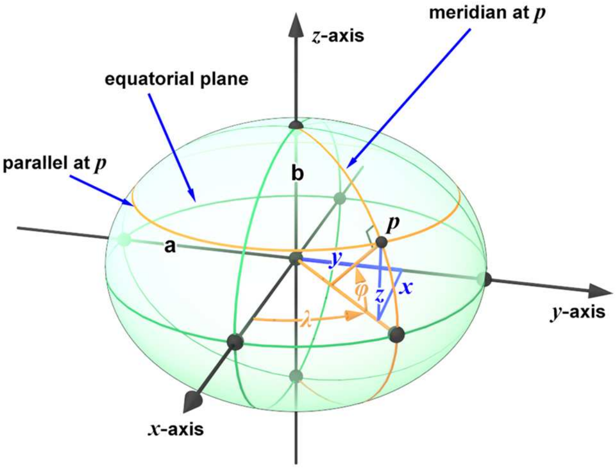


Figure 5.11 — Surface geodetic CS geometric and trigonometric relationships

EXAMPLE 4 The geodetic coordinate system for positions on the surface of an oblate ellipsoid is an example of a coordinate system of coordinate system type surface. The geometric and trigonometric relationships for the generating function of this coordinate system are illustrated in Figure 5.11. The generating function for this coordinate system depends on the major and minor semi-axis parameter values a and b. These are the coordinate system parameters. This coordinate system is fully specified in 5.3.8.18 as the Surface Geodetic coordinate system.

5.3.4 Coordinate-component surfaces and curves

5.3.4.1 Introduction

It is useful in some applications to reduce the dimensionality of a coordinate system by fixing the values of one or more of its coordinate-components. The resulting coordinate-component surfaces and curves are particularly useful when dealing with curvilinear coordinate systems. Thus, fixing the ellipsoidal height coordinate-component (h) of a 3D geodetic coordinate system to zero results in an induced surface geodetic coordinate system that represents positions on the surface of the ellipsoid modelling the Earth (see Figure 5.11). This International Standard provides functions for creating coordinate-component surfaces and curves, and specifies several standard induced surface coordinate systems.

The generating function of a 3D CS is a function of its three coordinate-components. Keeping one of the coordinate-components fixed (to a constant value) and varying the other two restricts the range of the generating function to a surface. This restricted generating function may be viewed as a surface CS generating function. Similarly, keeping two of the three coordinate-components fixed restricts the range of the generating function to a curve, and the restricted generating function may be viewed as a curve CS generating function. These observations motivate the definitions of coordinate-component surfaces and curves. The coordinate-component surface and coordinate-component curve concepts are required to specify induced CS relationships, to define the special coordinate curves parallel and meridian, and to define CS handedness. Coordinate-component curves are also used to define localized frames.

5.3.4.2 Coordinate-component surfaces and induced surface CSs

A *coordinate-component surface* is the surface that results from fixing the value of one of the three coordinate-components of a 3D CS while letting the values of the other two coordinate-components vary. A coordinate-component surface is identified by the ordinal number (*1st, 2nd, or 3rd*) of the coordinate-component that is fixed.

If $\boldsymbol{G}$ is the generating function of a 3D CS, and $\boldsymbol{c} = (u_0, v_0, w_0)$ is in the interior of the CS domain D, the three coordinate-component surfaces at $\boldsymbol{c}$ are defined by:

$\boldsymbol{S}_1[\boldsymbol{G}]((v,\ w)) = \boldsymbol{G}((u_0, v, w))$,
$\boldsymbol{S}_2[\boldsymbol{G}]((u,\ w)) = \boldsymbol{G}((u, v_0, w))$, and
$\boldsymbol{S}_3[\boldsymbol{G}]((u,\ v)) = \boldsymbol{G}((u, v, w_0))$.

The CS domain for $\boldsymbol{S}_1[\boldsymbol{G}]$ is the connected component of $\{(v,\ w) \text{ in } \mathbb{R}^2 |\ (u_0, v, w) \text{ in } D\}$ which contains $(v_0,\ w_0)$.
The CS domain for $\boldsymbol{S}_2[\boldsymbol{G}]$ is the connected component of $\{(u,\ w) \text{ in } \mathbb{R}^2 |\ (u, v_0, w) \text{ in } D\}$ which contains $(u_0,\ w_0)$.
The CS domain for $\boldsymbol{S}_3[\boldsymbol{G}]$ is the connected component of $\{(u,\ v) \text{ in } \mathbb{R}^2 |\ (u, v, w_0) \text{ in } D\}$ which contains $(u_0,\ v_0)$.

The CS ranges of these generating functions are, respectively, the 1st, 2nd, and 3rd coordinate-component surface at $\boldsymbol{c}$.

Each of these surface CSs shall be termed, respectively, the 1st, 2nd, and 3rd *induced surface CS* at $\boldsymbol{c}$.

EXAMPLE 1 Coordinate-component surface: The Geodetic 3D CS is specified in Table 5.14 with CS parameters a and b. The 3rd coordinate-component surface at coordinate $\boldsymbol{c} = (0,0,0)$ is the surface of an oblate ellipsoid with major semi-axis a and minor semi-axis b.
Induced surface CS: The Surface Geodetic CS specified in Table 5.24 is the 3rd induced surface CS for the Geodetic 3D CS at $\boldsymbol{c}$.

EXAMPLE 2 Coordinate-component surface: The Lococentric Cylindrical CS is specified in Table 5.17. The 3rd coordinate-component surface at coordinate $\boldsymbol{c} = (0,0,0)$ is a plane.
Induced surface CS: The Lococentric Surface Polar CS specified in Table 5.28 is the 3rd induced surface CS for the Lococentric Cylindrical CS at $\boldsymbol{c}$.

Several CSs of CS type surface that are specified in this standard are each an induced surface CS for a corresponding 3D CS.

5.3.4.3 Coordinate-component curves

A *coordinate-component curve* is the curve that results from fixing the values of two of the three coordinate-components of a 3D CS while letting the value of the other coordinate-component vary, or from fixing the value of one of the two coordinate-components of a surface or 2D CS while letting the value of the other coordinate-component vary. A coordinate-component curve is identified by the ordinal number (1st, 2nd, or in the 3D case 3rd) of the coordinate-component that varies.

The CS type 3D case:

If $\boldsymbol{G}$ is the generating function of a 3D CS, and $\boldsymbol{c} = (u_0,\ v_0,\ w_0)$ is in the interior of the CS domain D, then the three coordinate-component curves *at* $\boldsymbol{c}$ are defined by:

$\boldsymbol{C}_1[\boldsymbol{G}](u) = \boldsymbol{G}((u, v_0, w_0))$,
$\boldsymbol{C}_2[\boldsymbol{G}](v) = \boldsymbol{G}((u_0, v, w_0))$, and
$\boldsymbol{C}_3[\boldsymbol{G}](w) = \boldsymbol{G}((u_0, v_0, w))$.

The CS domain for $\boldsymbol{C}_1[\boldsymbol{G}]$ is the connected component of $\{u \text{ in } \mathbb{R} |\ (u, v_0, w_0) \text{ in } D\}$ which contains u_0.
The CS domain for $\boldsymbol{C}_2[\boldsymbol{G}]$ is the connected component of $\{v \text{ in } \mathbb{R} |\ (u_0, v, w_0) \text{ in } D\}$ which contains v_0.
The CS domain for $\boldsymbol{C}_3[\boldsymbol{G}]$ is the connected component of $\{w \text{ in } \mathbb{R} |\ (u_0, v_0, w) \text{ in } D\}$ which contains w_0.

The CS ranges of these functions are, respectively, the 1st, 2nd, and 3rd coordinate-component curve at $\boldsymbol{c}$.

NOTE The intersection of two coordinate-component surfaces at $\boldsymbol{c}$ is (the locus of) a coordinate-component curve: $\boldsymbol{C}_1 = \boldsymbol{S}_2 \cap \boldsymbol{S}_3,\ \boldsymbol{C}_2 = \boldsymbol{S}_1 \cap \boldsymbol{S}_3,\ \boldsymbol{C}_3 = \boldsymbol{S}_1 \cap \boldsymbol{S}_2$.

The CS type surface and CS type 2D cases:

If $\boldsymbol{G}$ is the generating function of a surface CS or 2D CS, and $\boldsymbol{c} = (u_0,\ v_0)$ is in the interior of the CS domain D, then the two coordinate-component curves at $\boldsymbol{c}$ are defined by:

$\boldsymbol{C}_1[\boldsymbol{G}](u) = \boldsymbol{G}\big((u, v_0)\big)$, and

$\boldsymbol{C}_2[\boldsymbol{G}](v) = \boldsymbol{G}\big((u_0, v)\big)$.

The CS domain for $\boldsymbol{C}_1[\boldsymbol{G}]$ is the connected component of $\{u$ in $\mathbb{R}|\ (u, v_0)$ in $D\}$ which contains u_0.
The CS domain for $\boldsymbol{C}_2[\boldsymbol{G}]$ is the connected component of $\{v$ in $\mathbb{R}|\ (u_0, v)$ in $D\}$ which contains v_0.

The CS ranges of these functions are, respectively, the *1st and 2nd* coordinate-component curve *at* $\boldsymbol{c}$.

EXAMPLE If $\boldsymbol{c} = (\rho, \theta_0)$ is in the interior of the CS domain of the Polar CS (5.3.3 Example 2), then the first coordinate-component curve is $\boldsymbol{C}_1[\boldsymbol{G}](\theta) = \boldsymbol{G}\left((\rho_0, \theta)\right) = \left[\rho_0 \cos\theta, \rho_0 \sin\theta\right]^{\mathrm{T}}$, and the 2nd coordinate-component curve is $\boldsymbol{C}_2[\boldsymbol{G}](\theta) = \boldsymbol{G}\big((\rho, \theta_0)\big) = [\rho\cos\theta_0, \rho\sin\theta_0]^{\mathrm{T}}$.

If $\boldsymbol{G}$ is the generating function for the Geodetic 3D CS or the Surface Geodetic CS, and $\boldsymbol{c} = (\lambda_0, \varphi_0, 0)$ in the 3D case or $\boldsymbol{c} = (\lambda_0, \varphi_0)$ in the surface case, then (see Figure 5.12):

a) the 1st coordinate-component curve at $\boldsymbol{c}$ shall be termed the *parallel* at $\boldsymbol{c}$, and

b) the 2nd coordinate-component curve at $\boldsymbol{c}$ shall be termed the *meridian*[11] at $\boldsymbol{c}$.

The meridian at $\boldsymbol{c} = (0,0,0)$ or $(0,0)$ shall be termed the *prime meridian*[12].

The parallel at $\boldsymbol{c} = (0,0,0)$ or $(0,0)$ shall be termed the *equator*.

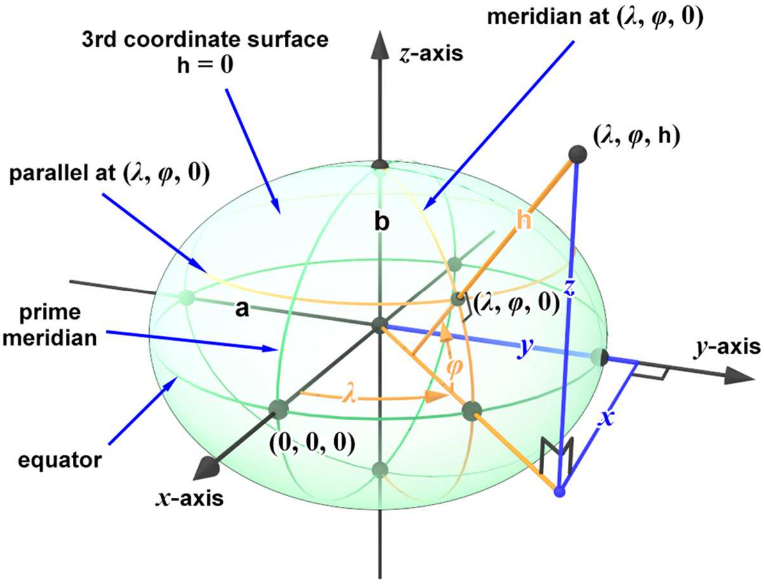


Figure 5.12 — Geodetic 3D CS geometry, and coordinate-component surface and curves

11 ISO 19111 defines the term meridian as "the intersection between an ellipsoid and a plane containing the shortest axis of the ellipsoid".

12 ISO 19111 defines the term prime meridian as "the meridian from which the longitudes of other meridians are quantified". In Clause 7, most, but not all, oblate ellipsoid Earth object reference models associate the Greenwich meridian with the prime meridian (see 7.4.5).

5.3.5 CS properties

5.3.5.1 Linearity

A CS with generating function $\boldsymbol{G}$ is a *linear CS* if the CS domain is all of $\mathbb{R}^n$ and $\boldsymbol{G}$ is a linear or affine function with respect to the vector space structure of $\mathbb{R}^n$ and position-space.

A *curvilinear CS* is a non-linear CS. In particular, Geodetic 3D CS and Surface Geodetic CS (Tables 5.14 and 5.24) and every map projection or augmented map projection CS (see 5.3.7) are curvilinear.

5.3.5.2 Orthogonality

A 3D CS, surface CS, or 2D CS with generating function $\boldsymbol{G}$ is *orthogonal* if the angle between any two coordinate-component curves at $\boldsymbol{p} = \boldsymbol{G}(\boldsymbol{c})$ is a right angle when $\boldsymbol{c}$ is any coordinate in the interior of the CS domain.

EXAMPLE The Polar CS of 5.3.3 EXAMPLE 2 is an orthogonal CS of type 2D. In this example the CS domain contains the coordinate (0,0) as a boundary point.

5.3.5.3 Linear CS properties: Cartesian, and orthogonal

In a linear CS, the k^{th} coordinate-component curve is a straight line. The k^{th} coordinate-component curve at the origin $\boldsymbol{q} = \boldsymbol{G}(\boldsymbol{0})$ of a linear CS is the *k^{th}-axis* where $\boldsymbol{0}$ is the all zero coordinate-component n-tuple in $\mathbb{R}^n$.

In a linear CS, if the angles between coordinate-component curves at any point are (pairwise) right angles, then that is the case at all points. In particular, a linear CS is orthogonal[13] if the axes are orthogonal.

A linear CS that is also orthogonal is a *Cartesian* CS[14].

EXAMPLE The Lococentric Euclidean 3D CS specified in Table 5.9 is a Cartesian CS since it is a linear CS and its coordinate-component curves intersect at right angles.

5.3.5.4 CS right-handedness and coordinate-component ordering

Given a 3D CS and a coordinate $\boldsymbol{c} = (u_0, v_0, w_0)$ in the interior of the CS domain, the coordinate-component curves $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, and $\boldsymbol{C}_3$ at $\boldsymbol{p} = \boldsymbol{G}(\boldsymbol{c})$ determine an ordered set of three tangent vectors:

$$\boldsymbol{t}_1 = \left.\frac{d\boldsymbol{C}_1}{du}\right|_{u=u_0},$$
$$\boldsymbol{t}_2 = \left.\frac{d\boldsymbol{C}_2}{dv}\right|_{v=v_0}, \text{ and}$$
$$\boldsymbol{t}_3 = \left.\frac{d\boldsymbol{C}_3}{dw}\right|_{w=w_0}.$$

An orthogonal 3D CS is a *right-handed CS* if for any coordinate $\boldsymbol{c}$ in the interior of the CS domain, the ordered set of tangent vectors $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, and $\boldsymbol{t}_3$ form a right-handed coordinate system as defined in IS0 80000-2. The right-handed CS property is determined, in part, by the order of the coordinate-components in the coordinate 3-

[13] Some publications use “rectangular” to denote an orthogonal linear CS, and “oblique” to denote a non-orthogonal linear CS.

[14] ISO 19111 defines Cartesian coordinate system as “a coordinate system that gives the position of points relative to n mutually-perpendicular axes”.

tuple. The order of the coordinate-components in the specification of an orthogonal 3D CS shall be restricted to an ordering that ensures a right-handed CS. This restriction is required for uniform treatment of directions, rotations, and orientations (see Clause 6 and 10.5).

The coordinate-component ordering in the specification of a surface CS that is induced on a coordinate-component surface of a 3D CS, shall use the coordinate-component order of the inducing 3D CS.

EXAMPLE 1 The Geodetic 3D CS (Table 5.14) coordinate-component ordering (λ, φ, h) ensures that the CS is right-handed. A similar ordering for the Planetodetic 3D CS (Table 5.15) is not right-handed because the tangent to Planetodetic longitude points opposite to the direction of the tangent to Geodetic longitude. Instead, the coordinate-component ordering (φ, λ, h) is specified to satisfy the right-handed CS requirement.

EXAMPLE 2 The Surface planetodetic CS (Table 5.25) coordinate-component ordering (φ, λ) is determined by the coordinate-component ordering (φ, λ, h) of the Planetodetic 3D CS (Table 5.15) which induces the Surface planetodetic CS as its 3rd coordinate-component surface.

5.3.6 CS localization

5.3.6.1 Introduction

Many applications need to perform operations involving multiple related coordinate systems. The relationships between coordinate systems used in an application often reflect corresponding relationships among the objects of interest within the application context. This International Standard provides a set of parameterized localization operators that can be used to translate and/or rotate a coordinate system within position-space. It is useful in some applications to reduce the dimensionality of a 3D coordinate system by fixing the values of one or more of its coordinate-components, resulting in an induced surface coordinate system (5.3.4). These two mechanisms can be combined to create localized induced surface coordinate systems.

5.3.6.2 Localization operators

A coordinate system can be translated and/or rotated to realize a local variant. A localization operation is used to accomplish this. The generating function of the original CS is composed with an appropriate localization operator to specify the generating function of the local variant CS. This method of specifying a local variant CS is termed *CS localization*. The result is termed a localized CS.

Three parameterized operators, termed *localization operators,* that operate on or between position-spaces are defined in Table 5.2. The inverses of these operators are defined in Table 5.3. The vectors in these tables, termed *localization parameters*, are vectors in (the range) position-space where $\boldsymbol{q}$ denotes the local origin (lococentre), and unit vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ denote the primary and secondary axis directions respectively. The localization operators require $\boldsymbol{r}$ and $\boldsymbol{s}$ to be *orthonormal vectors*, that is, $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthogonal ($\boldsymbol{r} \bullet \boldsymbol{s} = 0$) and normal ($\|\boldsymbol{r}\| = \|\boldsymbol{s}\| = 1$).

Table 5.2 — Localization operators

Localization operator	Domain	Range	Localization parameters	Operator definition
$\boldsymbol{L}_{3D}$	$\mathbb{R}^3$	$\mathbb{R}^3$	$\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, in $\mathbb{R}^3$ $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal	$\boldsymbol{L}_{3D}([x,y,z]^{\mathsf{T}}) = \boldsymbol{q} + x\boldsymbol{r} + y\boldsymbol{s} + z\boldsymbol{t}$, where: $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$.
$\boldsymbol{L}_{\text{Surface}}$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, in $\mathbb{R}^3$ $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal	$\boldsymbol{L}_{\text{Surface}}([x,y]^{\mathsf{T}}) = \boldsymbol{q} + x\boldsymbol{r} + y$
$\boldsymbol{L}_{2D}$	$\mathbb{R}^2$	$\mathbb{R}^2$	$\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, in $\mathbb{R}^2$ $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal	$\boldsymbol{L}_{2D}([x,y]^{\mathsf{T}}) = \boldsymbol{q} + x\boldsymbol{r} + y\boldsymbol{s}$

Table 5.3 — Localization inverse operators

Localization operator	Inverse operator definition
$\boldsymbol{L}_{3\mathrm{D}}$	$\boldsymbol{L}_{3\mathrm{D}}^{-1}(\boldsymbol{p}) = \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{r}\big)\boldsymbol{e}_1 + \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{s}\big)\boldsymbol{e}_2 + \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{t}\big)\boldsymbol{e}_3$
$\boldsymbol{L}_{\mathrm{Surface}}$	$\boldsymbol{L}_{\mathrm{Surface}}^{-1}(\boldsymbol{p}) = \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{r}\big)\boldsymbol{e}_1 + \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{s}\big)\boldsymbol{e}_2$
$\boldsymbol{L}_{2\mathrm{D}}$	$\boldsymbol{L}_{2\mathrm{D}}^{-1}(\boldsymbol{p}) = \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{r}\big)\boldsymbol{e}_1 + \big((\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{s}\big)\boldsymbol{e}_2$

There are several forms of CS localization depending on CS type and localization operator. A 3D or surface CS with generating function $\boldsymbol{G}$ is localized by composing $\boldsymbol{G}$ with the $\boldsymbol{L}_{3\mathrm{D}}$ localization operator. The localized CS is of the same CS type (CS type 3D or CS type surface, respectively). Its generating function is $\boldsymbol{G}_L \equiv \boldsymbol{L}_{3\mathrm{D}} \circ \boldsymbol{G}$ and has the same CS domain as $\boldsymbol{G}$.

There are two localization operators for a 2D CS. One uses localization parameters in $\mathbb{R}^3$ and produces a surface CS. The other uses localization parameters in $\mathbb{R}^2$ and produces a 2D CS.

a) A 2D CS with generating function $\boldsymbol{G}$ is localized by composing $\boldsymbol{G}$ with the $\boldsymbol{L}_{\mathrm{Surface}}$ localization operator. The localized CS is a surface CS. Its generating function is $\boldsymbol{G}_L \equiv \boldsymbol{L}_{\mathrm{Surface}} \circ \boldsymbol{G}$ and has the same CS domain as $\boldsymbol{G}$.

b) A 2D CS with generating function $\boldsymbol{G}$ is localized by composing $\boldsymbol{G}$ with the $\boldsymbol{L}_{2D}$ localization operator. The localized CS is a 2D CS. Its generating function is $\boldsymbol{G}_L \equiv \boldsymbol{L}_{2\mathrm{D}} \circ \boldsymbol{G}$ and has the same CS domain as $\boldsymbol{G}$.

The localization operator parameter $\boldsymbol{q}$ shall be termed the *lococentre*. A localized CS may be termed a *lococentric CS*.

The CS generated by $\boldsymbol{G}_{\mathrm{L}}$ is related to the original CS in that the geometry of the coordinate curves and surfaces are similar to that generated with $\boldsymbol{G}$ but with the geometry shifted over in position-space with a translation by $\boldsymbol{q}$ and oriented with respect to the $\boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$ basis vectors instead of the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ basis vectors. This use of localization operators specifies a family of CSs with generating functions $\boldsymbol{G}_{\mathrm{L}}$ parameterized by the localization parameters $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$. This employment of lococentric operators is used below in the specification of all the CSs that have "LOCOCENTRIC_" as a name prefix.

NOTE 1 CS localization preserves the following CS properties: linear/curvilinear, orthogonal, and Cartesian.

The relationship between a CS type and its localized version(s) is summarized in Table 5.4.

Table 5.4 — Localized CS type relationships

CS type	Localization operator	Resulting lococentric CS type
3D	$\boldsymbol{L}_{3\mathrm{D}}$	3D
Surface	$\boldsymbol{L}_{3\mathrm{D}}$	Surface
2D	$\boldsymbol{L}_{\mathrm{Surface}}$	
	$\boldsymbol{L}_{2\mathrm{D}}$	2D

NOTE 2 In the case of 3D position-space, a set of localization parameters $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ together with $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$ specifiy an orthonormal frame with $\boldsymbol{q}$ as its origin and $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ as its basis vectors (see 5.2.3). In effect, a localization operator transforms the geometry of the CS surfaces and coordinate-component curves with respect to the canonical frame of position-space to the same geometry with respect to the $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$ orthonormal frame by translating the position-space origin to $\boldsymbol{q}$ and rotating the axes to align with $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$.

For creating a localized induced surface CS, the operations of localization and inducing a surface CS (see 5.3.4.2) are order-independent in the following way.

Consider X, a 3D CS with generating function G_{3D}. For a given set of localization parameters, and a given value n =1, 2, or 3, a localized induced surface CS at coordinate $\boldsymbol{c}$ may be derived from X by two methods.

In one method, first X induces SX, the n^{th} induced surface CS for X at $\boldsymbol{c}$ with generating function $S_n[G_{\text{3D}}]$. Then SX is localized to LSX, a surface CS with generating function $L_{\text{Surface}} \circ S_n[G_{\text{3D}}]$.

In the other method, first X is localized to LX, a 3D CS with generating function $L_{\text{3D}} \circ G_{\text{3D}}$. Then LX induces SLX, the n^{th} induced surface CS for LX at $\boldsymbol{c}$ with generating function $S_n[L_{\text{3D}} \circ G_{\text{3D}}]$.

The two resulting localized surface CSs, LSX and SLX, are identical with generating functions $L_{\text{Surface}} \circ S_n[G_{\text{3D}}] = S_n[L_{\text{3D}} \circ G_{\text{3D}}]$ (see Figure 5.13). For examples, see Table 5.26 Note 1, Table 5.27 Note 2 and Table 5.28 Note 2.

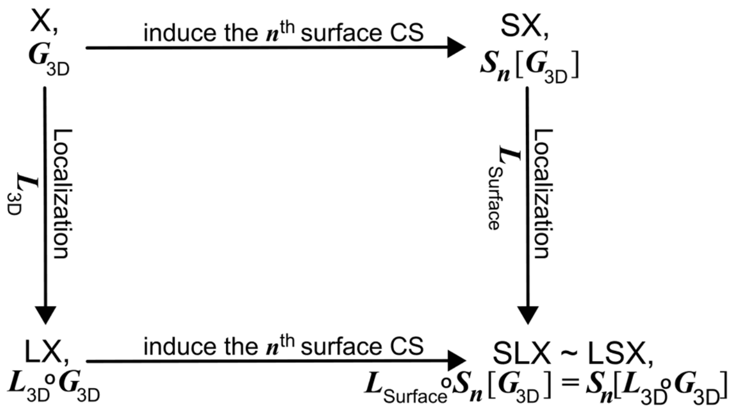


Figure 5.13 — Induced surface CS with localization

5.3.6.3 Localized frame and local tangent frame at a coordinate

The localization parameters $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ together with $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$ specifiy an orthonormal frame with $\boldsymbol{q}$ as its origin and $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ as its basis vectors (see 5.2.6.2 Note 2). Conversely, given an orthonormal frame within position-space, the orthonormal frame origin and it's first two basis vectors may be used as localization parameters, $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ respectively. The choice of these parameters and/or an orthonormal frame is often dependent on the intended application.

A 3D Cartesian CS with generating function $\boldsymbol{G}$ has an intrinsic vector space structure. Given a coordinate $\boldsymbol{c}$ and two mutually perpendicular vectors, with coordinates $\boldsymbol{c}_r$ and $\boldsymbol{c}_s$, an orthonormal frame may be specified with position-space vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$, where:

$$\boldsymbol{q} = \boldsymbol{G}(\boldsymbol{c}),$$
$$\boldsymbol{r} = \big(\boldsymbol{G}(\boldsymbol{c}_r) - \boldsymbol{G}(\boldsymbol{0})\big)/\big\|\big(\boldsymbol{G}(\boldsymbol{c}_r) - \boldsymbol{G}(\boldsymbol{0})\big)\big\|,$$
$$\boldsymbol{s} = \big(\boldsymbol{G}(\boldsymbol{c}_s) - \boldsymbol{G}(\boldsymbol{0})\big)/\big\|\big(\boldsymbol{G}(\boldsymbol{c}_s) - \boldsymbol{G}(\boldsymbol{0})\big)\big\|, \text{ and}$$
$$\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}.$$

An orthonormal frame constructed in this way is termed the *localized frame at coordinate* $\boldsymbol{c}$ *with parameters* $\boldsymbol{c}_r$ and $\boldsymbol{c}_s$, and the CS is termed the *reference CS of the localized frame*.

The axis directions in a localized frame are not required to be parallel to any of the reference CS axes, thus a localized frame has all the generality of an orthonormal frame while providing the ability to specify the required position-space vectors using a Cartesian CS directly. With restricted axis directions, any orthogonal 3D or surface CS may be used to specify an orthonormal frame.

EXAMPLE 1 Consider the case of a point $\boldsymbol{q}$ on an ellipsoid in which an orthonormal frame is desired which will have the surface point as its origin, the first basis vector $\boldsymbol{r}$ points in the direction of increasing longitude along the parallel at $\boldsymbol{q}$ (local east), the second basis vector $\boldsymbol{s}$ points in the direction of increasing latitude along the meridian at $\boldsymbol{q}$ (local north), and the third basis vector $\boldsymbol{t}$ points away from the interior of the ellipsoid in the direction perpendicular to the surface at $\boldsymbol{q}$ (local up) (see Figure 5.14). In this case, the Geodetic 3D CS with the ellipsoid semi-axis parameters provides a convenient way of computing the orthonormal frame vectors. If $\boldsymbol{c} = (\lambda, \varphi, 0)$ is the longitude, latitude, and height coordinate of point $\boldsymbol{q} = \boldsymbol{G}_{\text{Geodetic}}(\boldsymbol{c})$, the vectors $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ are computed as the normalised tangent vectors on the 1st, 2nd, and 3rd coordinate-component curves, respectively, at $\boldsymbol{c}$.

The ability to use the generating function of orthogonal CSs to compute vectors aligned to the CS coordinate curves at a point motivates the definition of a type of orthonormal frame whose primary and secondary axes are tangent to a coordinate surface.

If $\boldsymbol{G}_{3\text{D}}$ is the generating function of an orthogonal right-handed 3D CS and $\boldsymbol{c} = (u_0, v_0, w_0)$ is a coordinate in the interior of the CS domain, an orthonormal frame is specified by the position space vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$ where:

$\boldsymbol{q} = \boldsymbol{G}_{3\text{D}}(\boldsymbol{c})$ is the frame origin,
$\boldsymbol{r} = {}^{\boldsymbol{t}_1}/_{\|\boldsymbol{t}_1\|}$,
$\boldsymbol{s} = {}^{\boldsymbol{t}_2}/_{\|\boldsymbol{t}_2\|}$, and
$\boldsymbol{t} = {}^{\boldsymbol{t}_3}/_{\|\boldsymbol{t}_3\|}$,
where $\boldsymbol{t}_i, i = 1, 2, 3$ is the tangent vector on the i-th coordinate-component curve at $\boldsymbol{c}$ (see 5.3.5.4).

The orthogonal and right-handed properties of the CS implies that the normalized vectors $\boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$ form a right-handed orthonormal frame at origin point $\boldsymbol{q} = \boldsymbol{G}_{3\text{D}}(\boldsymbol{c})$ and that $\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$. The frame vectors $\boldsymbol{r}$, and $\boldsymbol{s}$ span a plane that is tangent at point $\boldsymbol{q}$ to the third coordinate surface $\boldsymbol{S}_3[\boldsymbol{G}_{3\text{D}}]\big((u, v)\big)$ at coordinate $\boldsymbol{c}$. This frame is termed the *local tangent frame* at coordinate $\boldsymbol{c}$, the CS is termed the *reference CS of the local tangent frame*, and $\boldsymbol{r}$, and $\boldsymbol{s}$ are the *local tangent vectors at* $\boldsymbol{c}$. A local tangent frame is a specialized form of localized frame.

This notion extends to orthogonal surfaces CSs. If $\boldsymbol{G}_{\text{Surface}}$ is the generating function of an orthogonal surface CS and $\boldsymbol{c} = (u_0, v_0)$ is a coordinate in the interior of the CS domain, a local tangent frame is specified by the position space vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$ where:

$\boldsymbol{q} = \boldsymbol{G}_{\text{Surface}}(\boldsymbol{c})$ is the frame origin,
$\boldsymbol{r} = {}^{\boldsymbol{t}_1}/_{\|\boldsymbol{t}_1\|}$,
$\boldsymbol{s} = {}^{\boldsymbol{t}_2}/_{\|\boldsymbol{t}_2\|}$, and
$\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s}$,
where $\boldsymbol{t}_i, i = 1, 2$, is the tangent vector on the i-th coordinate-component curves at $\boldsymbol{c}$.

The span of the vectors $\boldsymbol{r}$, and $\boldsymbol{s}$ is the tangent plane to the CS surface at point $\boldsymbol{q} = \boldsymbol{G}_{\text{Surface}}(\boldsymbol{c})$. Similarly, the CS is termed the reference CS for the local tangent frame, and $\boldsymbol{r}$, and $\boldsymbol{s}$ are the local tangent vectors at $\boldsymbol{c}$.

In the case of an induced surface CS where $\boldsymbol{G}_{\text{Surface}}\big((u, v)\big) = \boldsymbol{S}_3[\boldsymbol{G}_{3\text{D}}]\big((u, v)\big)$ and point $\boldsymbol{q} = \boldsymbol{G}_{3\text{D}}\big((u_0, v_0, w_0)\big)$ lies on the CS surface, the localized frame at $\boldsymbol{q}$ for $\boldsymbol{G}_{3\text{D}}$ and the local tangent frame at $\boldsymbol{q}$ for $\boldsymbol{G}_{\text{Surface}}$ coincide. In particular, $\boldsymbol{r} \times \boldsymbol{s} = \boldsymbol{t} = {}^{\boldsymbol{t}_3}/_{\|\boldsymbol{t}_3\|}$ because of the coordinate-component ordering restriction specified in 5.3.5.

The position-space vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$ that specify a local tangent frame may serve as the parameters for the localization operators. With this choice of localization parameters, a localized CS will be aligned to the coordinate curves of the reference CS at the lococentric origin $\boldsymbol{q}$ point. The specification of a local tangent frame at a coordinate only requires the coordinate value. The frame axis directions are restricted to the coordinate curve tangent directions computed from the CS generating function.

EXAMPLE 2 For the surface geodetic CS, the local tangent plane at coordinate $\boldsymbol{c} = (\lambda_0, \varphi_0)$, $\lambda_0 \neq \pm\pi$, is the orthonormal frame with origin $\boldsymbol{q} = \boldsymbol{G}_{\text{SurfGD}}(\boldsymbol{c})$ and basis vectors $, \boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$ where:

$$\boldsymbol{r} = \begin{bmatrix} -\sin\lambda_0 \\ \cos\lambda_0 \\ 0 \end{bmatrix}, \quad \boldsymbol{s} = \begin{bmatrix} -\cos\lambda_0 \sin\varphi_0 \\ -\sin\lambda_0 \sin\varphi_0 \\ \cos\varphi_0 \end{bmatrix}, \text{ and } \boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{s} = \begin{bmatrix} \cos\lambda_0 \cos\varphi_0 \\ \sin\lambda_0 \cos\varphi_0 \\ \sin\varphi_0 \end{bmatrix}$$

Localizing the Euclidean 3D CS with these vectors and localization operator $\boldsymbol{L}_{3\text{D}}$ produces the Lococentric Euclidean 3D CS with CS origin at $\boldsymbol{q}$, and CS axes aligned with vectors $\boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$. (See Figure 5.14.)

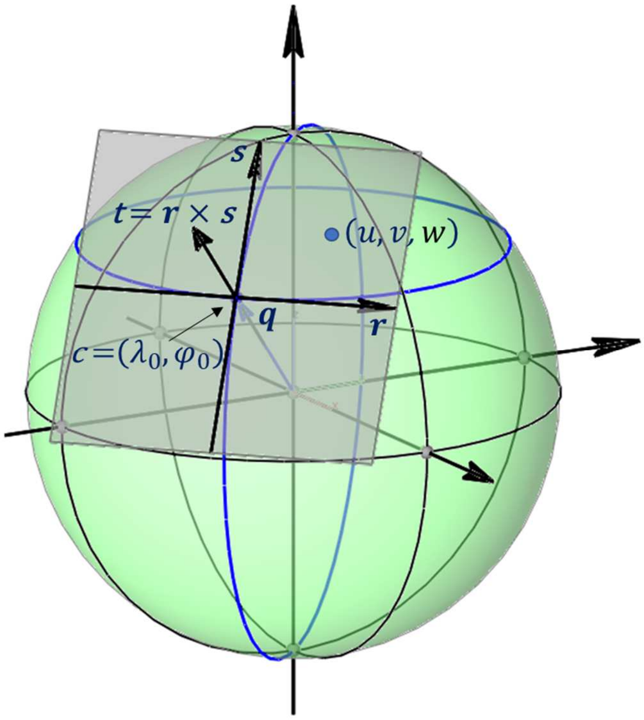


Figure 5.14 — Lococentric Euclidean CS on a local tangent plane

EXAMPLE 3 Using the vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, and $\boldsymbol{t}$ from Example 2 as localization parameters to localize the Azimuthal Spherical CS with the localization operator $\boldsymbol{L}_{3\text{D}}$ produces the Lococentric Azimuthal Spherical CS. (See Figure 5.15.)

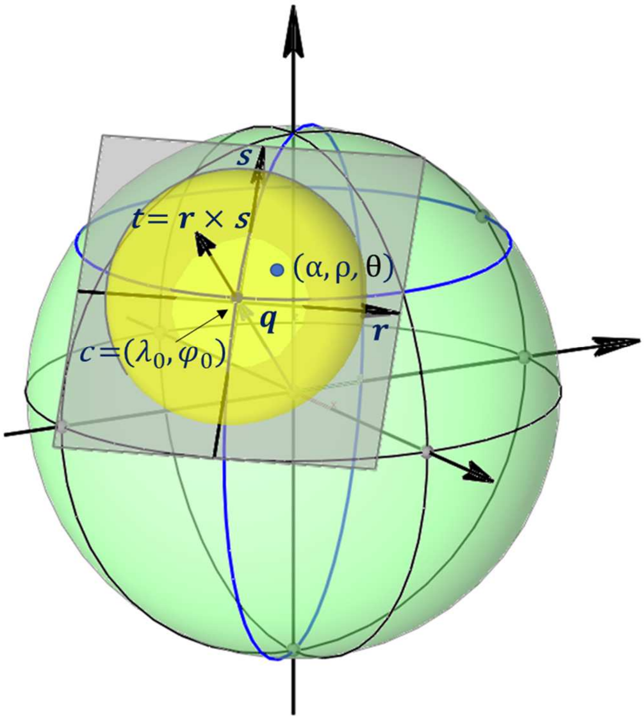


Figure 5.15 — Lococentric azimuthal spherical CS on a local tangent plane

The origin and basis vectors of an orthonormal frame can also be specified using the Cartesian vector space of a given local tangent frame at a coordinate. In some applications, this is a useful way of specification when the desired orthonormal frame is a rotation and/or a displacement of the given local tangent frame.

EXAMPLE 4 A local tangent frame at geodetic 3D coordinate $(\lambda, \varphi, 0)$ specifies frame vectors $\boldsymbol{q} = \boldsymbol{G}\big((\lambda, \varphi, 0)\big)$, $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{t}$. A new orthonormal frame is specified by frame origin $\tilde{\boldsymbol{q}}$ and basis vectors $\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{s}}$, and $\tilde{\boldsymbol{t}}$, where $\tilde{\boldsymbol{q}} = \boldsymbol{q} + h\boldsymbol{t}$ is the origin displaced by h units in the $\boldsymbol{t}$ direction, and $\tilde{\boldsymbol{r}} = \boldsymbol{R}\boldsymbol{r}$ and $\tilde{\boldsymbol{s}} = \boldsymbol{R}\boldsymbol{s}$ are the vectors $\boldsymbol{r}$, and $\boldsymbol{s}$ rotated about the $\boldsymbol{t}$-axis through an angle θ by rotation matrix $\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\tilde{\boldsymbol{t}} = \boldsymbol{t}$.

5.3.6.4 Vectors, directions and localized frames

Specification of vectors, including directions and vector quantities, requires an underlying vector space. In this standard, a vector is specified by a Cartesian coordinate in a designated localized frame (see 5.3.6.3) termed the *vector reference frame*. A direction is specified by a unit vector in a vector reference frame. A vector quantity is specified by a vector with a magnitude in a vector reference frame. This is addressed in greater detail in 8.4.5.

EXAMPLE The local up direction at geodetic coordinate $\boldsymbol{c} = (\lambda_0, \varphi_0)$, $\lambda_0 \neq \pm\pi$, is specified by the vector $[0 \quad 0 \quad 1]^{\mathrm{T}}$ with respect to the vector reference frame at coordinate $\boldsymbol{c}$.

5.3.7 Map projection coordinate systems

5.3.7.1 Map projections

Map projections are 2D models of a 3D curved surface. In this International Standard, map projections are limited to the surface of an oblate ellipsoid (including the special case of a sphere). A *map projection* (MP) is comprised of

a) an MP domain in the surface of an oblate ellipsoid,

b) a generating projection, and

c) an MP range in 2D coordinate-space,

where:

a) the MP domain is a connected subset of the surface of the oblate ellipsoid,

b) the MP range is a connected replete set, and

c) the *generating projection* is one-to-one from the MP domain in the oblate ellipsoid onto its MP range and its inverse function is smooth in the MP range interior.

NOTE 1 This definition may be generalized to any ellipsoid including tri-axial ellipsoids, but this International Standard only addresses map projections for oblate ellipsoids.

NOTE 2 The domain of a map projection is always a proper subset of the oblate ellipsoid surface. In particular, the domain of the Mercator map projection (see Table 5.18) omits the pole points.

The generating projection $\boldsymbol{P}$ is specified in terms of Surface Geodetic CS coordinates (see Table 5.24). The component functions P_1 and P_2 of the generating projection $\boldsymbol{P}$ shall be termed the *mapping equations*:

$$\boldsymbol{P}(\lambda, \varphi) = (u, v)$$

where:

$$u = P_1(\lambda, \varphi)$$
$$v = P_2(\lambda, \varphi).$$

The MP range coordinate-components u and v shall be termed *easting* and *northing*, respectively. The positive direction of the u-axis (the easting axis) shall be termed *map-east*. The positive direction of the v-axis (the northing axis) shall be termed *map-north*.

The inverse mapping equations are the component functions Q_1 and Q_2 of the inverse generating projection $\boldsymbol{Q} = \boldsymbol{P}^{-1}$:

$$\lambda = Q_1(u, v)$$
$$\varphi = Q_2(u, v)$$

5.3.7.2 Map projection as a surface CS

If the inverse generating projection of a map projection $\boldsymbol{Q}$ is composed with the Surface Geodetic CS generating function $\boldsymbol{G}_{\text{GD}}$, the resulting function $\boldsymbol{G}_{\text{MP}} = \boldsymbol{G}_{\text{GD}} \circ \boldsymbol{Q}$ is the generating function of a surface CS (see Figure 5.16). The CS domain is the MP range. In this International Standard, a *map projection CS* shall be a surface CS for which the generating function is implicitly specified in terms of the mapping equations of a map projection.

In some cases, the Surface Geodetic coordinates with coordinate-component $\varphi = \pm\pi/2$ are not in the MP domain nor are they in the range of $\boldsymbol{Q}$. However, if the composite function $\boldsymbol{G}_{\text{MP}}^{-1} = \boldsymbol{P} \circ \boldsymbol{G}_{\text{GD}}^{-1}$ is continuous at the pole points $(0, 0, \pm b)$, then $\boldsymbol{G}_{\text{MP}}$ and $\boldsymbol{G}_{\text{MP}}^{-1}$ shall be extended by continuity to include the pole points in the CS range.

NOTE The CS generating function $\boldsymbol{G}_{\text{MP}} = \boldsymbol{G}_{\text{GD}} \circ \boldsymbol{Q}$ is not to be confused with the generating projection $\boldsymbol{P}$.

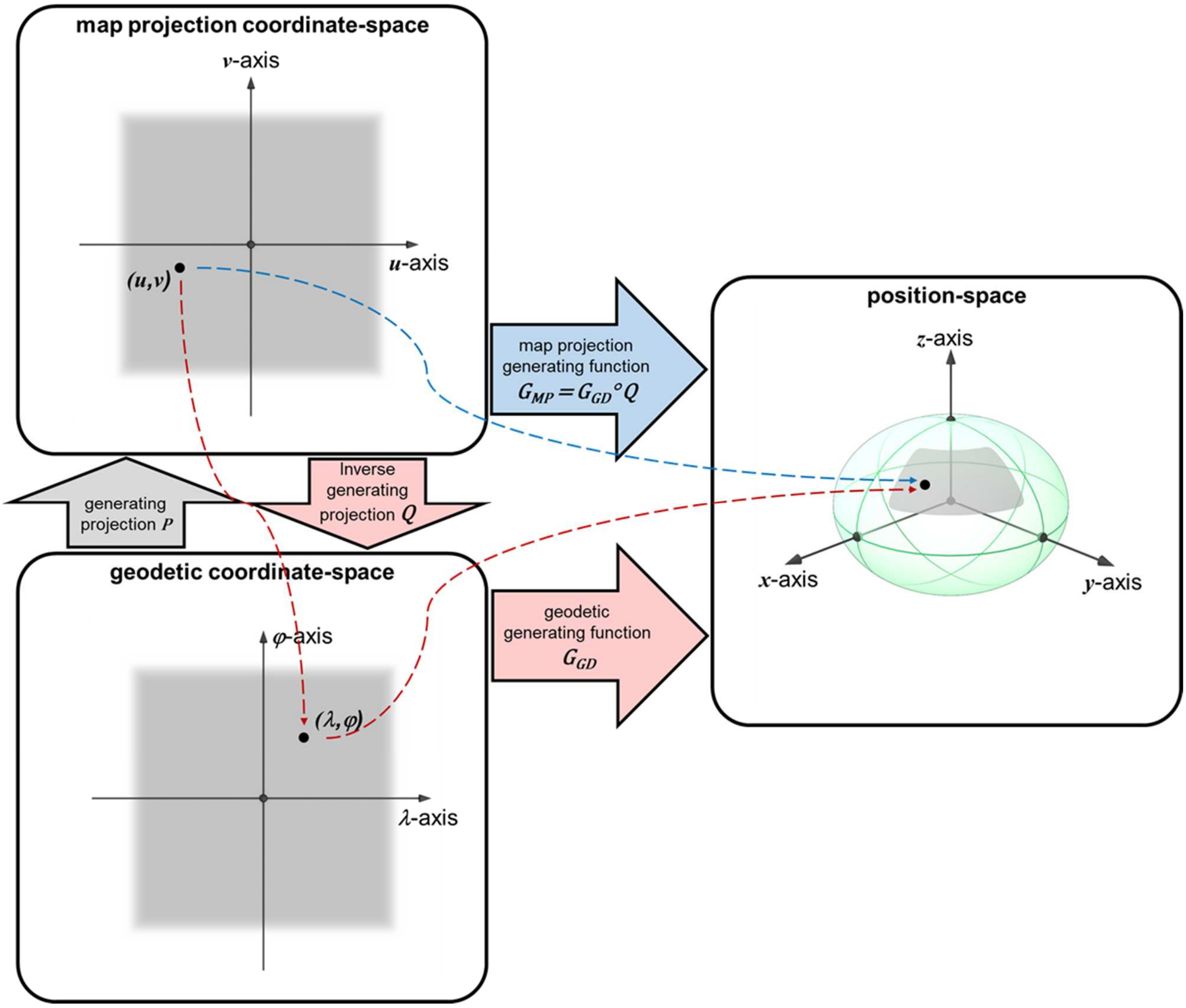


Figure 5.16 — A map projection generating function

5.3.7.3 Map projection geometry

5.3.7.3.1 Introduction

A map projection CS is a curvilinear[15] CS of type surface. In general, lengths, angles, or areas in 2D coordinate-space have no direct corelation to the geometry of the surface formed by a surface CS in position-space. In particular, the Euclidean distance between a pair of Surface Geodetic coordinates has no obvious meaning in position-space. In contrast, map projections are specifically designed so that coordinate-space geometry will model one or more geometric aspects of the corresponding oblate ellipsoid surface in position-space.

[15] For some map projections (see Mercator and Equidistant Cylindrical), coordinate curves project to coordinate-space in a rectilinear grid of straight lines. In position-space these curves lie on the ellipsoid surface.

The map projection CSs specified in this International Standard are based on formulations so that one or more geometric aspects of the MP domain in the oblate ellipsoid surface are approximated or modelled by the corresponding aspect in coordinate-space. If distance is to be modelled, the length of the line segment between two map coordinates is related approximately or exactly to the length of the corresponding surface curve. Similarly, one or more of directions, areas, the angles between two intersecting curves, and shapes may be related approximately or exactly to the corresponding geometric aspect on the oblate ellipsoid surface.

The extent to which these aspects are or are not closely related is an indication of distortion. Some map projection CSs are designed to eliminate distortion for one geometric aspect (such as angles or area). Others are designed to reduce distortion for several geometric aspects. In general, distortion tends to increase with the size of the oblate ellipsoid MP domain relative to the total oblate ellipsoid surface area. Conversely, distortion errors may be reduced by restricting the size of the MP domain. Map projections specified in this International Standard in the context of a spatial reference frame may have areas of definition beyond which the projection should not be used for some application domains due to unacceptable distortion[16].

5.3.7.3.2 Conformal map projections

A *conformal map projection* preserves angles. For such map projections, when two surface curves on an oblate ellipsoid meet at the angle α, the image of those curves in the map coordinate-space meet at the same angle α [THOM].

In addition, [THOM] contains a derivation based on the theory of complex variables to obtain conditions that specify when a projection is conformal. The map projections specified in Table 5.18 through Table 5.22 are conformal. The Equidistant Cylindrical MP specified in Table 5.23 is not conformal.

NOTE The conformal property is local in that a conformal map projection preserves angles at a point but does not necessarily preserve shape or area. In particular, a large projected triangle may appear distorted under a conformal map projection.

5.3.7.3.3 Point distortion

One indicator of map projection length distortion is the ratio of lengths between an infinitesimal line segment in coordinate-space and the corresponding curve in position-space. Given a point in the interior of the MP range with Surface Geodetic coordinate (λ,φ) the *directional point distortion*[17] at (λ,φ) with respect to a smooth surface curve passing through the point is the ratio of the differential distance in coordinate-space to the differential arc length at (λ,φ) along the curve as determined by the mapping equations.

The *latitudinal point distortion* at (λ,φ), denoted $j(\lambda,\varphi)$, is the directional point distortion with respect to the meridian at (λ,φ). It is computed in the direction of the meridian at the point as:

$$j(\lambda,\varphi) = \lim_{\Delta\to 0} \frac{\Delta(\text{arc length in coordinate space})}{\Delta(\text{arc length along a meridian})} = \frac{\sqrt{(\partial u/\partial\varphi)^2 + (\partial v/\partial\varphi)^2}}{\mathcal{M}(\varphi)}$$

where $\mathcal{M}(\varphi)$ is the radius of curvature in the meridian as specified in Table 5.6.

The *longitudinal point distortion* at (λ,φ), denoted $k(\lambda,\varphi)$, is the directional point distortion with respect to the parallel at (λ,φ). It is computed in the direction of the parallel at the point as:

---

16 It is a consequence of the *Theorema Egregium* of Gauss that no map projection CS can eliminate all distortion.

17 This concept is found in the literature under a variety of names. The term "point distortion" is introduced to avoid ambiguity.

$$k(\lambda, \varphi) = \lim_{\Delta \to 0} \frac{\Delta(\text{arc length in coordinate space})}{\Delta(\text{arc length along a parallel})} = \frac{\sqrt{(\partial u/\partial \varphi)^2 + (\partial v/\partial \varphi)^2}}{\mathcal{N}(\varphi)\cos(\varphi)}$$

where $\mathcal{N}(\varphi)$ is the radius of curvature in the prime vertical as specified in Table 5.6.

If a map projection is conformal, then the directional point distortion is independent of the direction of the curve at the point. In particular, $j(\lambda,\varphi) = k(\lambda,\varphi)$ for conformal map projections.

It is common practice in cartography to convert map projection coordinate-space to a display coordinate-space by means of a scaling factor. The scaling factor σ shall be termed a *map scale* [HTDP] and a point in the display space shall be termed a *display coordinate*[18]. The relationship of a display coordinate (u_d, v_d) to a map coordinate (u, v) is:

$$u_d = \sigma u$$
$$v_d = \sigma v$$

Map scale is commonly expressed as a ratio 1:n.

EXAMPLE A map scale printed on a map sheet as 1:50 000 corresponds to $\sigma = 1/50\ 000$.

For a conformal map projection, the infinitesimal ratio of display distance to arc length along a parallel is the *point scale* at (λ,φ) and is denoted by k_{scaled}. The relationship between point scale and point distortion is:

$$k_{\text{scaled}}(\lambda,\varphi) = \sigma\, k(\lambda,\varphi).$$

5.3.7.3.4 Geodetic azimuth and map azimuth

The *geodetic azimuth*[19] from a non-polar point $\boldsymbol{p}_1$ on the surface of an ellipsoid to a second point $\boldsymbol{p}_2$ on the surface is the curve azimuth (A.7.1.3) at $\boldsymbol{p}_1$ of the shortest geodesic curve segment (10.7) connecting $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ (see Figure 5.17). The range of azimuth values α shall be $0 \leq \alpha < 2\pi$. The definition and range constraints apply to points in both hemispheres.

In a map projection CS, the *map azimuth* from a coordinate $\boldsymbol{c}_1$ to a coordinate $\boldsymbol{c}_2$ is defined as the angle from the v-axis (map-north) clockwise to the line segment connecting $\boldsymbol{c}_1$ to $\boldsymbol{c}_2$. In general, the map azimuth for a pair of coordinates will differ in value from the geodetic azimuth of the corresponding points on the oblate ellipsoid.

[18] The distinction between a map projection coordinate and a display coordinate is not usually made explicit in the literature. The term “display coordinate” is introduced to avoid ambiguity.

[19] More general definitions that allow measurements of azimuth angle clockwise or counterclockwise and from the north or south side of the meridian are in use. The generalization to the case for which one or more of the two points is not on the surface is treated in [RAPP1] and [RAPP2]. The more general definitions are not required for subsequent SRM concepts.

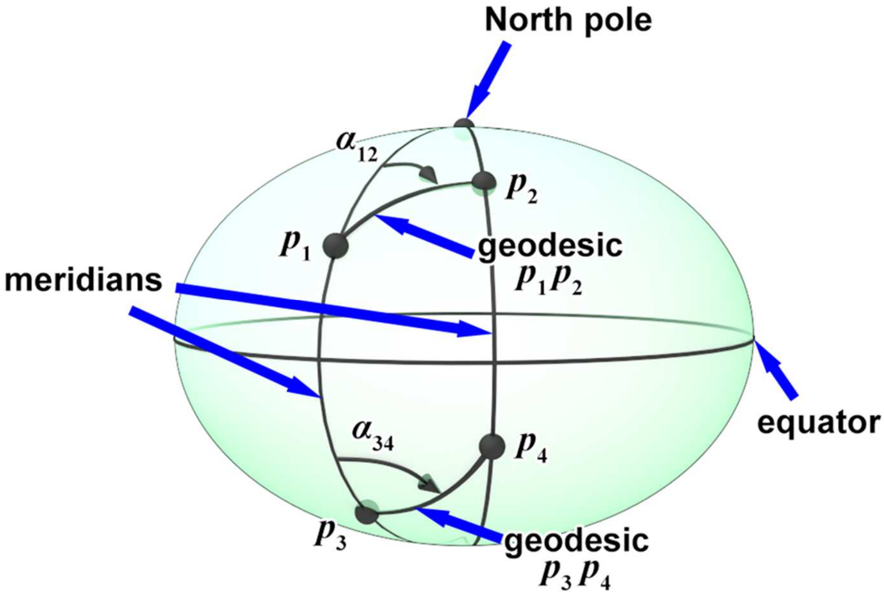


Figure 5.17 — Geodetic azimuths α_{12} from p_1 to p_2 and α_{34} from p_3 to p_4

5.3.7.3.5 Convergence of the meridian

Given a point (λ,φ) in the interior of the MP domain, the meridian through that point is projected to a curve in coordinate-space that passes through the corresponding coordinate. The angle γ at the coordinate in the clockwise direction from the curve to the v-axis (map-north) direction shall be termed the *convergence of the meridian* (COM).

The relationship $\gamma(\lambda,\varphi) = \text{arctan2}\left(-\frac{\partial u}{\partial \varphi}, \frac{\partial v}{\partial \varphi}\right)$ is used to derive the formulae for COM from the mapping equations of each of the map projections[20]. The COM angle is adjusted to the range $-\pi < \gamma \leq \pi$.

NOTE 1 If the map projection is conformal, then an equivalent relationship is given by: $\gamma(\lambda,\varphi) = \text{arctan2}\left(\frac{\partial v}{\partial \lambda}, \frac{\partial u}{\partial \lambda}\right)$.

20 The function arctan2 is defined in A.8.1.

A typical geometry illustrating the COM at a point $\boldsymbol{p}$ is shown for the transverse Mercator map projection in Figure 5.18.

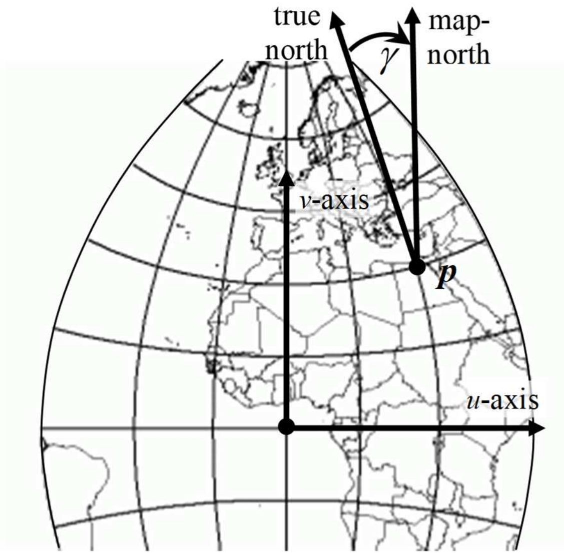


Figure 5.18 — Convergence of the meridian

NOTE 2 If $\boldsymbol{p}_2$ is a map coordinate directly map-north of a map coordinate $\boldsymbol{p}_1$ (it has a larger $\boldsymbol{v}$ coordinate-component), then the map azimuth is zero, but the geodetic azimuth will not, in general, be zero. The geodetic azimuth is approximately the sum of the map azimuth and the COM if the two points are sufficiently close together.

5.3.7.4 Relationship to projection functions

5.3.7.4.1 Projection functions

Projection functions are defined in A.9. In some cases, to realize a map projection CS, its generating projection is derived from a projection function. The derivation involves two steps. The first step is to restrict the domain of the projection function to a specified region of a given oblate ellipsoid so that the restricted function is one-to-one, ensuring that only a single point on the oblate ellipsoid maps to a given point in the projection. The range of a projection function is a surface in 3D position-space. The second step is to associate the surface in 3D position-space to a 2D coordinate-space without introducing additional distortions.

In the case of planar projection functions, including the orthographic, perspective, and stereographic projection functions, the range is in a plane that can be identified with 2D coordinate-space by selecting an origin and unit axis points.

In the case of the cylindrical and conic projection functions, the range surface is a cylinder or a cone, respectively. These surfaces are developable surfaces and, except for a line of discontinuity, are homoeomorphic to a subset of 2D coordinate-space with a homeomorphism that has a Jacobian determinant equal to one. Conceptually, these surfaces can be unwrapped to a flat plane without stretching the surface.

The Polar Stereographic MP (Table 5.22) is derived from the stereographic projection function, and in the spherical case, it is a conformal map projection. The same derivation may be applied to an oblate ellipsoid. However, the resulting map projection will not have the conformal property. For this reason, the generalization of the Polar Stereographic map projection mapping equations from the spherical case to the non-spherical oblate ellipsoid case is not derived from the polar stereographic projection function. Instead, it is derived analytically to preserve the conformal property. Similarly, the Mercator map projection (Table 5.18) is designed to have the conformal property and is not derived from the cylindrical projection function even in the case of a sphere.

EXAMPLE Polar Stereographic: Given a sphere with a polar point $\boldsymbol{p}$, the tangent plane to the sphere at $\boldsymbol{p}$ and the opposite polar point $\boldsymbol{v}$ specify a stereographic planar projection function $\boldsymbol{P}$ (see A.9.2.3). The restriction of $\boldsymbol{P}$ to a subsurface of the sphere that excludes $\boldsymbol{v}$, is the generating projection for the sphere case of a Polar Stereographic map projection. In Figure 5.19 the position $\boldsymbol{s}$ on the sphere is projected to point $\boldsymbol{t}$ on a plane.

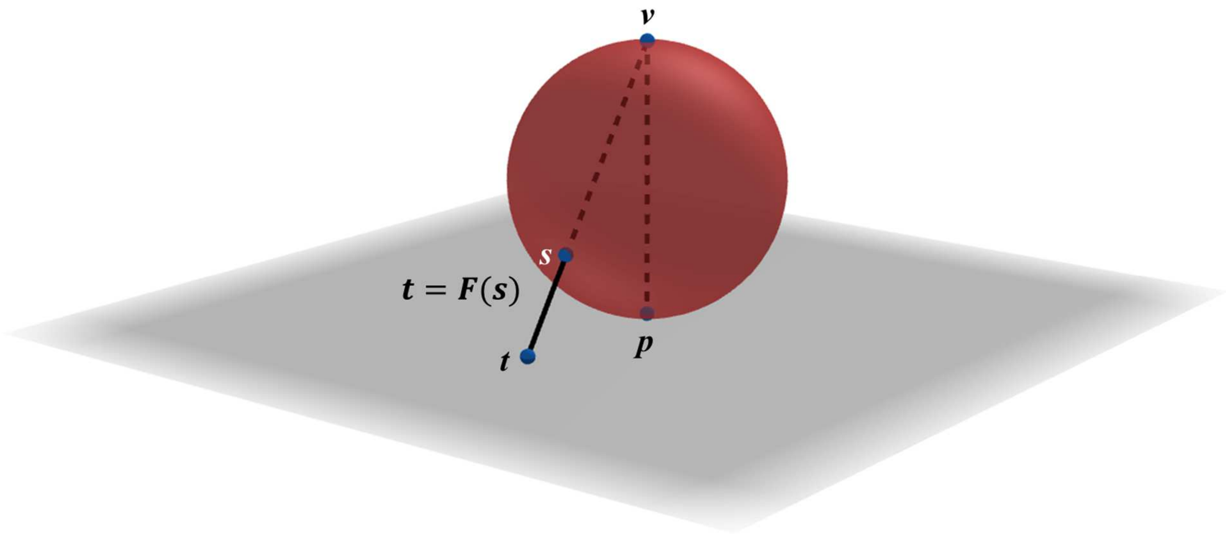


Figure 5.19 — Polar Stereographic map projection

5.3.7.4.2 Map projection classification

The use of projection functions to derive map projections with desirable properties is limited by their functional nature but does motivate some classifications of map projections derived by other means. These classifications include tangent and secant map projections as well as conic and cylindrical map projections [SNYD, p. 5].

5.3.7.4.3 Cylindrical map projections

A map projection is classified as *cylindrical* if:

a) all meridians of the oblate ellipsoid project to parallel straight lines that are equally spaced with respect to the longitude of the meridians, and

b) all parallels of the oblate ellipsoid project to parallel straight lines that are perpendicular to the meridian images.

As a consequence, the COM, $\gamma(\lambda, \varphi) = 0$ for a map projection of class cylindrical.

EXAMPLE The Mercator map projection (Table 5.18) and the equidistant cylindrical map projection (Table 5.23) are both examples of the cylindrical classification.

A cylindrical map projection is *tangent* if the longitudinal point distortion is equal to one along the equator. It is *secant* if the longitudinal point distortion is equal to one along two parallels equally spaced from the equator in latitude. In that case, the parallel with positive latitude shall be termed the *standard parallel*. Tangent and secant cylindrical map projections are illustrated in Figure 5.20.

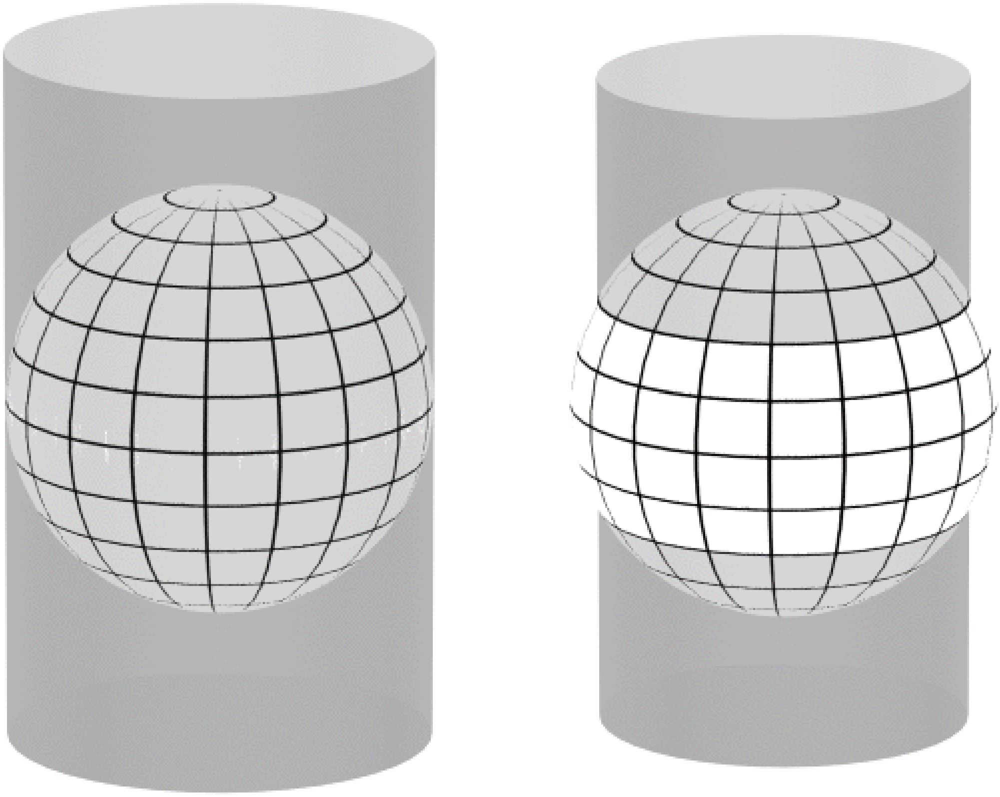


Figure 5.20 — Tangent and secant cylindrical map projections

5.3.7.4.4 Conic map projections

A map projection is classified as *conic* if:

a) all meridians of the oblate ellipsoid project to radial straight lines that are equally spaced in radial angle with respect to the longitude of the meridians, and

b) all parallels of the oblate ellipsoid project to concentric arcs that are perpendicular to the meridian images.

As a consequence, COM, γ depends only on λ for conic map projections because the projections of meridians are straight line segments with an angle depending only on the longitude.

EXAMPLE Lambert Conformal Conic (see Table 5.21) satisfies conditions a) and b) and thus is classified as a conic projection.

A conic map projection is *tangent* if along one parallel the longitudinal point distortion is equal to one. It is *secant* if the longitudinal point distortion is equal to one along two parallels that are not symmetric about the equator. In that case, the two parallels shall be termed the *standard parallels* (see Figure 5.21).

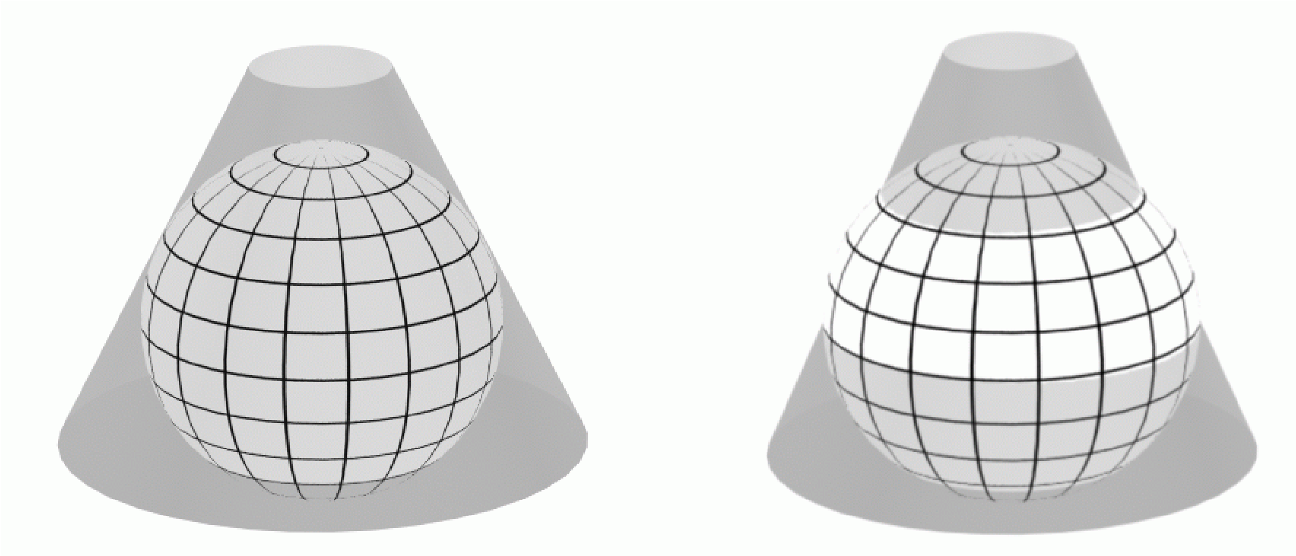


Figure 5.21 — Tangent and secant conical map projections

5.3.7.5 Map projection CS common parameters

To avoid negative coordinate-component values or to reduce the magnitude of the values in a region of interest in the coordinate-space of a map projection, two mapping equation parameters shall be provided to control the position of the coordinate-space origin $(0,0)$. One, denoted as u_{F}, shall be termed the *false easting* and shall offset easting values. The second, denoted as v_{F}, shall be termed the *false northing* and shall offset northing values.

A map projection CS specification may specify additional mapping equation parameters. The mapping equation parameters, including false easting and false northing, shall also be termed the CS parameters.

The CS parameters *longitude of origin*, denoted by $\lambda_{\mathrm{origin}}$, and *latitude of origin*, denoted by $\varphi_{\mathrm{origin}}$, are historically associated with some map projection CSs. Typically, the position with map coordinate $(u_{\mathrm{F}}, v_{\mathrm{F}})$ has geodetic longitude equal to $\lambda_{\mathrm{origin}}$ and/or geodetic latitude equal to $\varphi_{\mathrm{origin}}$.

If $\lambda_{\mathrm{origin}}$ is present as a CS parameter in a map projection CS specification, it is used as the longitudinal centring function parameter (see Λ_{C} in Table 5.6).

The CS parameter *central scale*, denoted by k_0, when present, is intended to control the tangent/secant characteristics of the map projection CS and is therefore close to, but does not generally exceed, 1,0. In the case of a sphere, $k_0 = 1$ corresponds to a tangent projection, and $k_0 < 1$ corresponds to a secant projection. Typically, some point in the MP domain with geodetic longitude equal to $\lambda_{\mathrm{origin}}$ and/or geodetic latitude equal to $\varphi_{\mathrm{origin}}$ will have a longitudinal point distortion equal to k_0.

For some MP cases, if $k_0 < 1$, there will exist a parallel with longitudinal point distortion equal to 1 at each point on the parallel. The latitude of such a parallel is termed a *secant latitude*, or a *latitude of true scale*.

NOTE Central scale should not be confused with Map scale. Map scales (see 5.3.7.3.3) are typically much smaller in magnitude and are applied directly to the coordinate-space. In particular, if a transverse Mercator map projection with central scale value $k_0 = 0{,}9996$ is to be scaled 1:50 000 on a map sheet display, then the mapping equations $P_1(\lambda, \varphi)$, and $P_2(\lambda, \varphi)$ are evaluated with $k_0 = 0{,}9996$ and the display coordinates $(\sigma P_1(\lambda, \varphi),\ \sigma P_2(\lambda, \varphi))$, are plotted on the map sheet with $\sigma = (1/50\ 000)$.

5.3.7.6 Augmented map projections

5.3.7.6.1 Augmentation with ellipsoidal height

A 3D CS can be specified from a map projection CS. The canonical embedding of a point (u, v) in $\mathbb{R}^2$ to the point $(u, v, 0)$ in the uv-plane of $\mathbb{R}^3$ allows map points in 2D coordinate-space to be augmented with a third coordinate axis, the w-axis of $\mathbb{R}^3$. To be considered as a 3D CS, an augmented 3-tuple (u, v, w) of coordinates in the *augmented map projection* coordinate-space shall be associated to a unique position in position-space. The association is to ellipsoidal height $h = w$. Given an augmented coordinate-tuple (u, v, w) for which (u, v) belongs to the coordinate range of the underlying generating projection, the associated position is given in Geodetic 3D coordinates (λ, φ, h) where (λ, φ) is projected to (u, v) by the map projection mapping equations. The third coordinate-space coordinate w is the vertical coordinate and the Geodetic 3D coordinate constraints on negative values of h impose corresponding constraints on allowed values for w. In some application domains, other vertical coordinate measures are used (see Clause 9). Augmentation is restricted to ellipsoidal height in this International Standard.

5.3.7.6.2 Distortion in augmented map projections

In addition to map projection distortion (see 5.3.7.3.3), augmentation causes additional distortion. Consider the two straight-line segments between the pairs of coordinate-space points $\{(u_1, v_1, 0), (u_2, v_2, 0)\}$ and $\{(u_1, v_1, w), (u_2, v_2, w)\}$ with $w > 0$ (see Figure 5.22). In augmented map projection geometry, the two line segments have the same length in coordinate-space. The corresponding curve in position-space of the first line segment is a surface curve of the oblate ellipsoid (or sphere). The corresponding second curve is outside of the oblate ellipsoid (or sphere) and has longer arc length than the first, and the length difference increases with w.

In general, vertical angles are not preserved and the angular error will vary with the point distortion value. These and other distortions have profound implications for dynamic equations that are beyond the scope of this International Standard

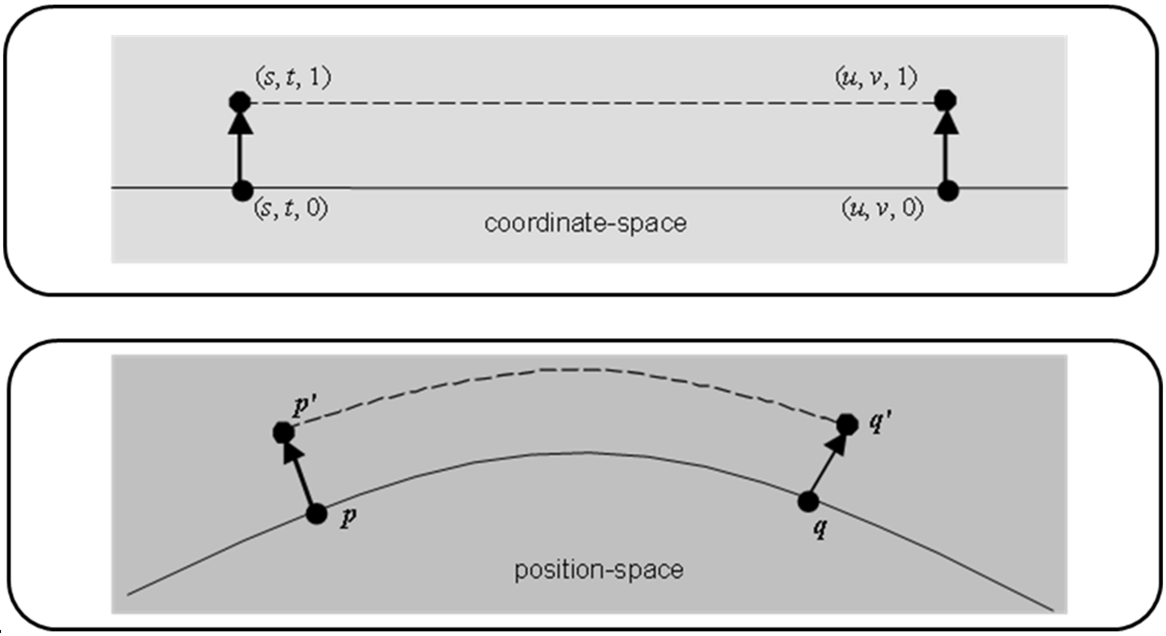


Figure 5.22 — Vertical distortion

5.3.8 CS specifications

5.3.8.1 Specification table elements and common functions and parameters

The CSs specified in this International Standard are presented in Table 5.8 through Table 5.37. Each CS specification specifies the values of all elements presented in Table 5.5.

Table 5.5 — Coordinate system specification elements

Element	Definition
Description	A description of the CS including a common name, if any.
CS label	The label of the CS (see 13.2.2).
CS code	The code of the CS (see 13.2.3). Code 0 (UNSPECIFIED) is reserved.
Function type	Either “generating function” or “map projection”.
CS descriptor	One of: 3D linear, 3D curvilinear, surface linear, surface curvilinear, map projection, 2D linear, 2D curvilinear, 1D linear, 1D curvilinear, or surface (map projection) and 3D (augmented map projection).
Properties	Either “none” or a list of one or more properties of the CS chosen from the following: orthogonal, Cartesian, and in the case of an MP, conformal, or not conformal.
CS parameters and constraints	The CS parameters (if any) along with any constraints on how those parameters interrelate, otherwise “none”.
Coordinate-components	Coordinate-component symbols and common names in a specified order.
CS Domain *-- or --* **MP Domain**	For Function type generating function: **CS Domain** For Function type map projection: **MP Domain**
Generating function *-- or --* **Mapping equations**	For Function type generating function: **Generating function** For Function type map projection: **Mapping equations**
Domain of the inverse	The domain of the inverse of the CS generating function or the domain of the inverse of the generating projection.
Inverse	The inverse of the CS generating function or the inverse of the generating projection.
COM	For map projection CSs, the equation for γ in radians. Otherwise "n/a".
Point distortion	For map projection CSs, the equation for k, if conformal, or the equations for k and j, if non-conformal. Otherwise "n/a".
Figures	Zero or more figure(s) that explain and illustrate the CS.
Notes	Optional, non-normative information concerning the CS, otherwise "none".
References	The references (see 13.2.5).

A specific ordering of coordinate-components for a coordinate n-tuple in a CS specification is required in this International Standard for clarity of presentation, to avoid ambiguity in the specification of the API, and, in the case of an orthogonal 3D CS, to ensure the right-handed CS property (see 5.3.5.4). Coordinate values may be represented using any of the methods delineated in 8.5.1.

Several specified CS generating functions and mapping equations and/or their inverses use some common intermediate functions or parameters associated with oblate ellipsoids. For clarity and concise presentation, these functions and parameters are defined in Table 5.6.

Table 5.6 — Common parameters and functions of an oblate ellipsoid

Function or parameter	Symbol and defining expression
major semi-axis	a
minor semi-axis	b
flattening	$f = 1 - \frac{b}{a}$
(first) eccentricity	$\varepsilon^2 = 1 - \left(\frac{b}{a}\right)^2$ alternative equivalent expression: $\varepsilon^2 = 2f - f^2$
second eccentricity	$(\varepsilon')^2 = \frac{\varepsilon^2}{(1-\varepsilon^2)}$
radius of curvature in the prime vertical	$\mathcal{N}(\varphi) = \frac{a}{\sqrt{1-\varepsilon^2 \sin^2 \varphi}}$
radius of curvature in the meridian	$\mathcal{M}(\varphi) = \frac{a(1-\varepsilon^2)}{\left(\sqrt{1-\varepsilon^2 \sin^2 \varphi}\right)^3}$ $= \frac{(1-\varepsilon^2)}{a^2}\left(\mathcal{N}(\varphi)\right)^3$
meridional distance to equator	$\mathcal{S}(\varphi) = \int_0^{\varphi} \mathcal{M}(\xi)\, d$
longitudinal centring about λ_C	$\Lambda_C(\lambda, \lambda_C) = \begin{cases} \lambda - \lambda_C & \text{if } -\pi < \lambda - \lambda_C \le \pi \\ \lambda - \lambda_C - 2\pi & \text{if } \pi < \lambda - \lambda_C \\ \lambda - \lambda_C + 2\pi & \text{if } \lambda - \lambda_C \le -\pi \end{cases}$

NOTE 1 Replacing λ_C with $-\lambda_C$ gives the inverse of the longitudinal centring function. That is:

if $\Lambda^* = \Lambda_C(\lambda,\ \lambda_C)$, then $\lambda = \Lambda_C(\Lambda^*,\ -\lambda_C)$.

NOTE 2 The function arctan2, used in many CS specification tables, is defined in A.8.1.

Table 5.7 presents a directory of the CS specifications in this International Standard. Each listed CS is specified in a separate table that is indicated by the hyperlink in the corresponding cell in the “Table number” column. Additional CSs may be specified by registration in accordance with Clause 13.

Table 5.7 — CS specification directory

Function type	CS type	Label and description	Table number

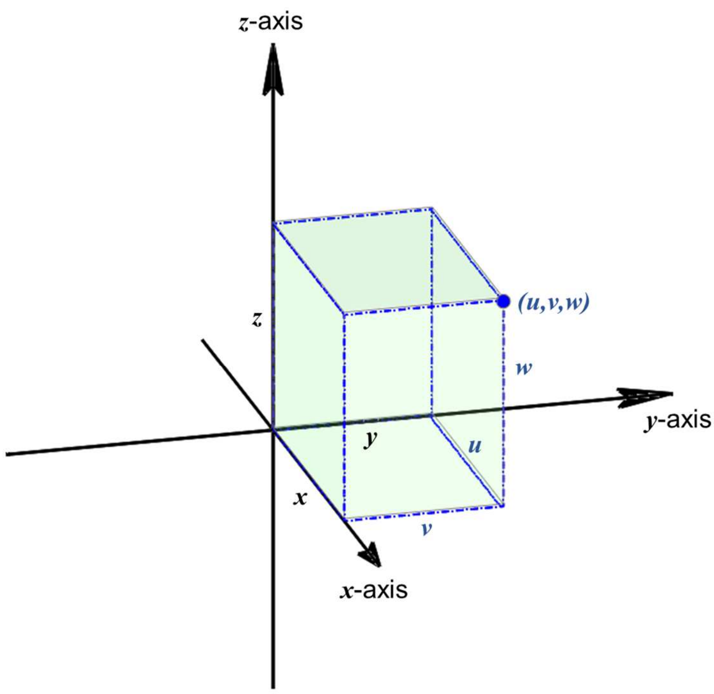
Function type	CS type	Label and description	Table number
Generating function	3D	EUCLIDEAN_3D Euclidean 3D	Table 5.8
		LOCOCENTRIC_EUCLIDEAN_3D Lococentric Euclidean 3D	Table 5.9
		EQUATORIAL_SPHERICAL Equatorial Spherical	Table 5.10
		LOCOCENTRIC_EQUATORIAL_SPHERICAL Lococentric Equatorial Spherical	Table 5.11
		AZIMUTHAL_SPHERICAL Azimuthal Spherical	Table 5.12
		LOCOCENTRIC_AZIMUTHAL_SPHERICAL Lococentric Azimuthal Spherical	Table 5.13
		AZIMUTHAL_CYLINDRICAL Azimuthal Cylindrical	Table 5.36
		LOCOCENTRIC_AZIMUTHAL_CYLINDRICAL Localized Azimuthal Cylindrical	Table 5.37
		GEODETIC Geodetic 3D	Table 5.14
		PLANETODETIC Planetodetic 3D	Table 5.15
		CYLINDRICAL Cylindrical	Table 5.16
		LOCOCENTRIC_CYLINDRICAL Lococentric Cylindrical	Table 5.17
Map projection	Surface and augmented 3D	MERCATOR Mercator	Table 5.18
		OBLIQUE_MERCATOR_SPHERICAL Oblique Mercator Spherical	Table 5.19
		TRANSVERSE_MERCATOR Transverse Mercator	Table 5.20
		LAMBERT_CONFORMAL_CONIC Lambert Conformal Conic	Table 5.21
		POLAR_STEREOGRAPHIC Polar Stereographic	Table 5.22
		EQUIDISTANT_CYLINDRICAL Equidistant Cylindrical	Table 5.23
Generating function	Surface	SURFACE_GEODETIC Surface Geodetic	Table 5.24
		SURFACE_PLANETODETIC Surface Planetodetic	Table 5.25
		LOCOCENTRIC_SURFACE_EUCLIDEAN Lococentric Surface Euclidean	Table 5.26

Function type	CS type	Label and description	Table number
		LOCOCENTRIC_SURFACE_AZIMUTHAL Lococentric Surface Azimuthal	Table 5.27
		LOCOCENTRIC_SURFACE_POLAR Lococentric Surface Polar	Table 5.28
	2D	EUCLIDEAN_2D Euclidean 2D	Table 5.29
		LOCOCENTRIC_EUCLIDEAN_2D Lococentric Euclidean 2D	Table 5.30
		AZIMUTHAL Azimuthal	Table 5.31
		LOCOCENTRIC_AZIMUTHAL Lococentric Azimuthal	Table 5.32
		POLAR Polar	Table 5.33
		LOCOCENTRIC_POLAR Lococentric Polar	Table 5.34
	1D	EUCLIDEAN_1D Euclidean 1D	Table 5.35

5.3.8.2 Euclidean 3D CS specification

Table 5.8 — Euclidean 3D CS

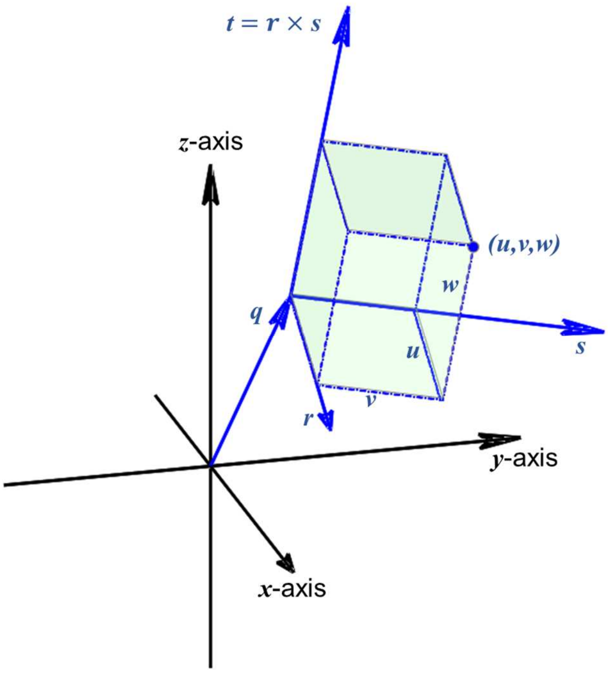
Element	Specification
Description	Euclidean 3D
CS label	EUCLIDEAN_3D
CS code	1
Function type	Generating function
CS descriptor	3D linear
Properties	Cartesian
CS parameters and constraints	none
Coordinate-components	$u,\ v,\ w$
CS Domain	$\mathbb{R}^3$
Generating function	$\boldsymbol{G}_{E3D}\big((u,v,w)\big) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}_{E3D}^{-1}\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = (u,\ v,\ w)$
COM	n/a

Element	Specification
Point distortion	n/a
Figures	
Notes	Coordinate-space 3-tuples are identified with position-space 3-tuples.
References	[EDM]

5.3.8.3 Lococentric Euclidean 3D CS specification

Table 5.9 — Lococentric Euclidean 3D CS

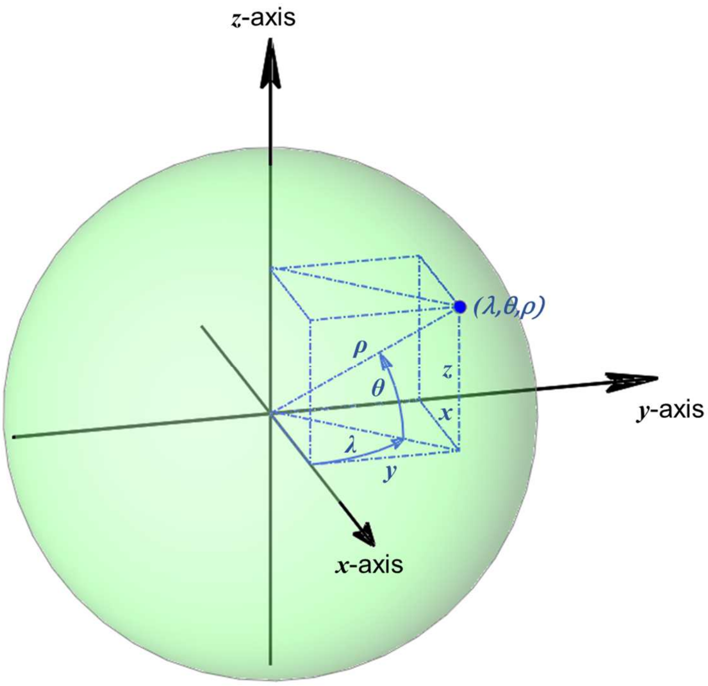
Element	Specification
Description	Localization of the Euclidean 3D CS
CS label	LOCOCENTRIC_EUCLIDEAN_3D
CS code	2
Function type	Generating function
CS descriptor	3D linear
Properties	Cartesian
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	u, v, w

Element	Specification
CS Domain	$\mathbb{R}^3$
Generating function	$\boldsymbol{G}_{\text{LE3D}}\big((u,v,w)\big) = \boldsymbol{L}_{\text{3D}} \circ \boldsymbol{G}_{\text{E3D}}\big((u,v,w)\big)$ $= [x,y,z]^{\top}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\ \ y = \boldsymbol{p} \bullet \boldsymbol{e}_2,\ \ z = \boldsymbol{p} \bullet \boldsymbol{e}_3,$ $\boldsymbol{p} = \ \boldsymbol{q} + u\,\boldsymbol{r} + v\,\boldsymbol{s} + w\,\boldsymbol{r} \times \boldsymbol{s},$ $\boldsymbol{L}_{\text{3D}} =$ the 3D localization operator, and $\boldsymbol{G}_{\text{E3D}} =$ the Euclidean 3D CS generating function
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}^{-1}([x,y,z]^{\top}) = \boldsymbol{G}_{\text{E3D}}^{-1} \circ \boldsymbol{L}_{\text{3D}}^{-1}\,([x,y,z]^{\top})$ $= (u,v,w)$ where: $u = \boldsymbol{p} \bullet \boldsymbol{r},\ \ \ v = \boldsymbol{p} \bullet \boldsymbol{s},\ \ \ w = \boldsymbol{p} \bullet (\boldsymbol{r} \times \boldsymbol{s}),$ $\boldsymbol{p} = \ (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3) - \boldsymbol{q},$ $\boldsymbol{L}_{\text{3D}}^{-1} =$ the inverse 3D localization operator, and $\boldsymbol{G}_{\text{E3D}}^{-1} =$ the Euclidean 3D CS inverse generating function
COM	n/a
Point distortion	n/a
Figures	
Notes	1) Euclidean 3D CS (see Table 5.8) is a special case with $\boldsymbol{q} = [0,0,0]$, $\boldsymbol{r} = [1,0,0]$, $\boldsymbol{s} = [0,1,0]$. 2) The generating function is the composition of the generating function for Euclidean 3D CS (see Table 5.8) with the 3D localization operator (see 5.3.6.2).
References	[EDM]

5.3.8.4 Equatorial Spherical CS specification

Table 5.10 — Equatorial Spherical CS

Element	Specification
Description	Equatorial Spherical
CS label	EQUATORIAL_SPHERICAL
CS code	3
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	none
Coordinate-components	λ: longitude in radians, θ: spherical latitude in radians, and ρ: radius.
CS Domain	$\left\{(\lambda,\theta,\rho) \text{ in } \mathbb{R}^3 \;\middle\vert\; -\pi < \lambda \le \pi \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \text{ and } 0 < \rho\right\}$ $\cup \left\{(0,\theta,\rho) \text{ in } \mathbb{R}^3 \;\middle\vert\; \theta = \pm\frac{\pi}{2} \text{ and } 0 < \rho\right\} \cup \{(0,0,0)\}$
Generating function	$\boldsymbol{G}_{\text{ES}}\big((\lambda,\theta,\rho)\big) = \begin{bmatrix} \rho\cos(\theta)\cos(\lambda) \\ \rho\cos(\theta)\sin(\lambda) \\ \rho\sin(\theta) \end{bmatrix}$
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}_{\text{ES}}^{-1}([x,y,z]^{\mathsf{T}}) = (\lambda,\theta,\rho)$, where: $\lambda = \operatorname{arctan2}(y,x)$, $\theta = \begin{cases} \arcsin\left(z/\rho\right) \text{ principal value}, & \text{if } \rho > 0 \\ 0 & \text{if } \rho = 0 \end{cases}$ and: $\rho = \sqrt{x^2+y^2+z^2}$.
COM	n/a
Point distortion	n/a

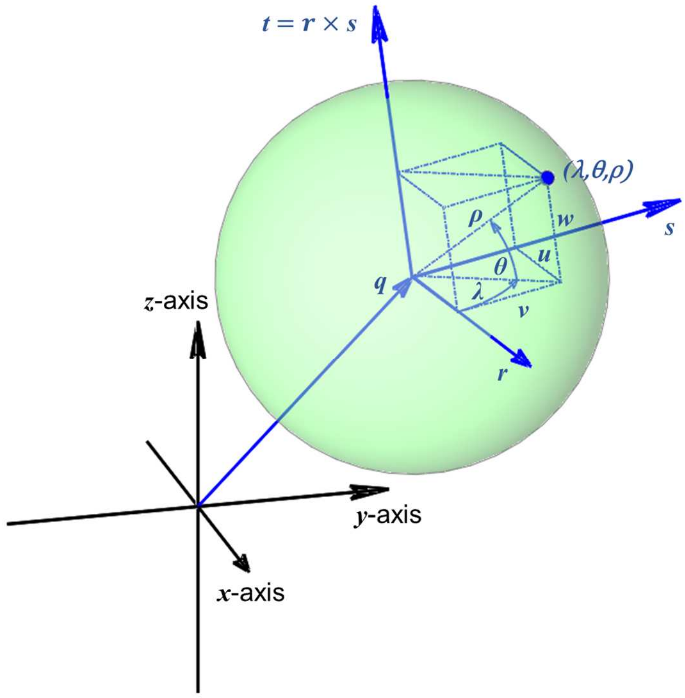
Element	Specification
Figures	
Notes	1) The Equatorial Spherical CS is not intrinsically associated with any specific sphere. 2) In many application domains, the co-latitude $\psi = \frac{\pi}{2} - \theta$ is used. The spherical latitude θ has been specified for compatibility with astronomical declination. The modifier "equatorial" is used to emphasize this difference. 3) The inverse generating function is discontinuous on the z-axis.
References	[HCP]

5.3.8.5 Lococentric Equatorial Spherical CS specification

Table 5.11 — Lococentric Equatorial Spherical CS

Element	Specification
Description	Localization of the Equatorial Spherical CS.
CS label	LOCOCENTRIC_EQUATORIAL_SPHERICAL
CS code	4
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal

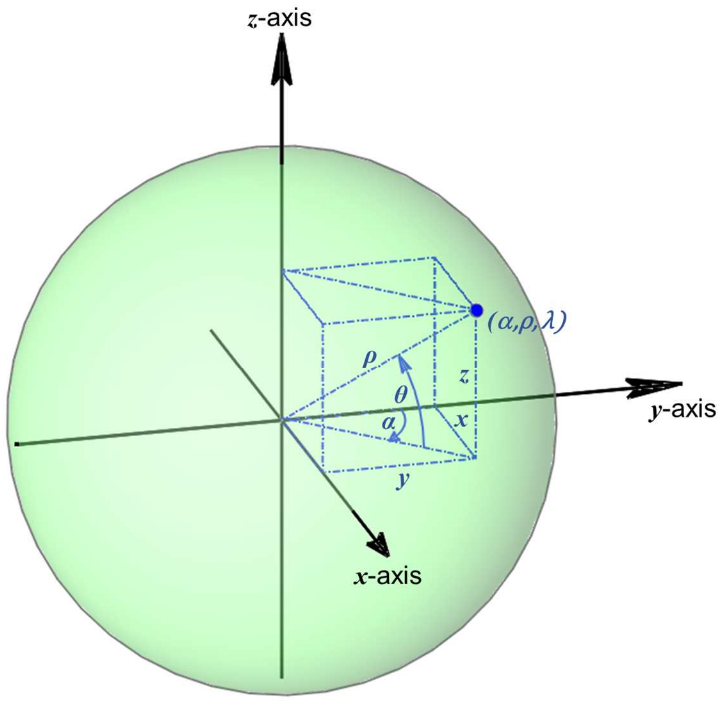
Element	Specification
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	λ: longitude in radians, θ: spherical latitude in radians, and ρ: radius.
CS Domain	$\left\{(\lambda,\theta,\rho) \text{ in } \mathbb{R}^3 \;\middle\vert\; -\pi < \lambda \leq \pi \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \text{ and } 0 < \rho\right\}$ $\cup \left\{(0,\theta,\rho) \text{ in } \mathbb{R}^3 \;\middle\vert\; \theta = \pm\frac{\pi}{2} \text{ and } 0 < \rho\right\} \cup \{(0,0,0)\}$
Generating function	$\boldsymbol{G}\big((\lambda,\theta,\rho)\big) = \boldsymbol{L}_{3\mathrm{D}} \circ \boldsymbol{G}_{\mathrm{ES}}\big((\lambda,\theta,\rho)\big)$ $= [x,y,z]^{\top}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\; y = \boldsymbol{p} \bullet \boldsymbol{e}_2,\; z = \boldsymbol{p} \bullet \boldsymbol{e}_3,$ $\boldsymbol{p} = \boldsymbol{q} + \rho(\cos(\theta)\,(\cos(\lambda)\,\boldsymbol{r} + \sin(\lambda)\,\boldsymbol{s}) + \sin(\theta)\,\boldsymbol{r} \times \boldsymbol{s}),$ $\boldsymbol{L}_{3\mathrm{D}} =$ the 3D localization operator, and $\boldsymbol{G}_{\mathrm{ES}} =$ the Equatorial Spherical CS generating function
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}^{-1}([x,y,z]^{\top}) = \boldsymbol{G}_{\mathrm{ES}}^{-1} \circ \boldsymbol{L}_{3\mathrm{D}}^{-1}([x,y,z]^{\top}) = (\lambda,\theta,\rho),$ where: $\lambda = \operatorname{arctan2}(v,u),$ $\theta = \begin{cases} \arcsin\left({}^{w}/_{\rho}\right) \text{ principal value,} & if\ \rho > 0 \\ 0 & if\ \rho = 0 \end{cases}$ $\rho = \sqrt{u^2 + v^2 + w^2},$ $u = \boldsymbol{p} \bullet \boldsymbol{r},\; v = \boldsymbol{p} \bullet \boldsymbol{s},\; w = \boldsymbol{p} \bullet (\boldsymbol{r} \times \boldsymbol{s}),$ $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3) - \boldsymbol{q},$ $\boldsymbol{L}_{3\mathrm{D}}^{-1} =$ the inverse 3D localization operator, and $\boldsymbol{G}_{\mathrm{ES}}^{-1} =$ the Equatorial Spherical CS inverse generating function
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	The generating function is the composition of the generating function for Equatorial Spherical CS (see Table 5.10) with the 3D localization operator (see 5.3.6.2).
References	[EDM]

5.3.8.6 Azimuthal Spherical CS specification

Table 5.12 — Azimuthal Spherical CS

Element	Specification
Description	Azimuthal Spherical
CS label	AZIMUTHAL_SPHERICAL
CS code	5
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	none
Coordinate-components	α: azimuth in radians, ρ: radius, and θ: depression/elevation angle in radians.
CS Domain	$\left\{(\alpha,\rho,\theta) \text{ in } \mathbb{R}^3 \;\middle|\; 0 \le \alpha < 2\pi \text{ and } 0 < \rho \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$ $\cup \left\{(0,\rho,\theta) \text{ in } \mathbb{R}^3 \;\middle|\; 0 < \rho \text{ and } \theta = \pm\frac{\pi}{2}\right\} \cup \{(0,0,0)\}$

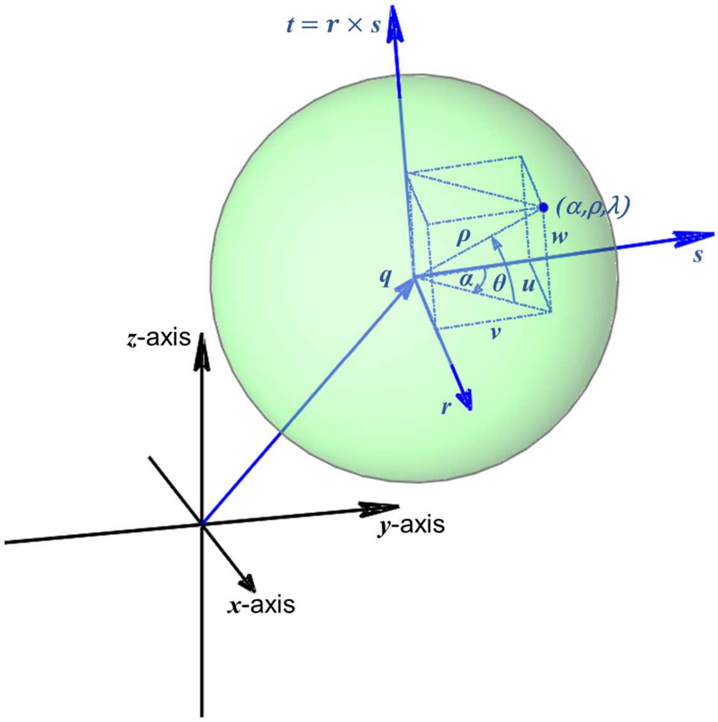
Element	Specification
Generating function	$\boldsymbol{G}_{\text{AS}}\big((\alpha,\rho,\theta)\big) = \begin{bmatrix} \rho\cos(\theta)\sin(\alpha) \\ \rho\cos(\theta)\cos(\alpha) \\ \rho\sin(\theta) \end{bmatrix}$
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}_{\text{AS}}^{-1}([x,y,z]^{\top}) = (\alpha,\rho,\theta)$, where: $\alpha = \begin{cases} \text{arctan2}(x,y) & \text{if } x \geq 0 \\ 2\pi + \text{arctan2}(x,y) & \text{if } x < 0 \end{cases}$ $\rho = \sqrt{x^2+y^2+z^2}$, and $\theta = \begin{cases} \arcsin\left({}^{z}/_{\rho}\right) \text{ principal value,} & \text{if } \rho > 0 \\ 0 & \text{if } \rho = 0. \end{cases}$
COM	n/a
Point distortion	n/a
Figures	
Notes	1) The inverse generating function is discontinuous on the z-axis. 2) The commonly used coordinate-component orderings are either *(ρ,α,θ)* or *(α,θ,ρ)*. The coordinate-component ordering has been specified as *(α,ρ,θ)* to ensure that this CS is right-handed. Compliant coordinate value representations are delineated in 8.5.1.
References	[EDM]

5.3.8.7 Lococentric Azimuthal Spherical CS specification

Table 5.13 — Lococentric Azimuthal Spherical CS

Element	Specification

Element	Specification
Description	Localization of the Azimuthal Spherical CS
CS label	LOCOCENTRIC_AZIMUTHAL_SPHERICAL
CS code	6
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	α: azimuth in radians, ρ: radius, and θ: depression/elevation angle in radians.
CS Domain	$\left\{(\alpha,\rho,\theta) \text{ in } \mathbb{R}^3 \;\middle|\; 0 \leq \alpha < 2\pi \text{ and } 0 < \rho \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$ $\cup \left\{(0,\rho,\theta) \text{ in } \mathbb{R}^3 \;\middle|\; 0 < \rho \text{ and } \theta = \pm\frac{\pi}{2}\right\} \cup \{(0,0,0)\}$
Generating function	$\boldsymbol{G}\big((\alpha,\rho,\theta)\big) = \boldsymbol{L}_{3\mathrm{D}} \circ \boldsymbol{G}_{\mathrm{AS}}\big((\alpha,\rho,\theta)\big)$ $= [x,y,z]^{\mathsf{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\ y = \boldsymbol{p} \bullet \boldsymbol{e}_2,\ z = \boldsymbol{p} \bullet \boldsymbol{e}_3$, $\boldsymbol{p} = \boldsymbol{q} + \rho(\cos(\theta)\,(\sin(\alpha)\,\boldsymbol{r} + \cos(\alpha)\,\boldsymbol{s}) + \sin(\theta)\,\boldsymbol{r} \times \boldsymbol{s})$, $\boldsymbol{L}_{3\mathrm{D}} =$ the 3D localization operator, and $\boldsymbol{G}_{\mathrm{AS}} =$ the Azimuthal Spherical CS generating function
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}^{-1}([x,y,z]^{\mathsf{T}}) = \boldsymbol{G}_{\mathrm{AS}}^{-1} \circ \boldsymbol{L}_{3\mathrm{D}}^{-1}([x,y,z]^{\mathsf{T}}) = (\alpha,\rho,\theta)$, where: $\alpha = \begin{cases} \operatorname{arctan2}(u,v) & \text{if } u \geq 0 \\ 2\pi + \operatorname{arctan2}(u,v) & \text{if } u < 0 \end{cases}$ $\theta = \begin{cases} \arcsin\left({}^{w}/_{\rho}\right) \text{ principal value}, & \text{if } \rho > 0 \\ 0, & \text{if } \rho = 0 \end{cases}$ $\rho = \sqrt{u^2 + v^2 + w^2}$ $u = \boldsymbol{p} \bullet \boldsymbol{r},\ \ v = \boldsymbol{p} \bullet \boldsymbol{s},\ \ w = \boldsymbol{p} \bullet (\boldsymbol{r} \times \boldsymbol{s})$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3) - \boldsymbol{q}$, $\boldsymbol{L}_{3\mathrm{D}}^{-1} =$ the inverse 3D localization operator, and $\boldsymbol{G}_{\mathrm{AS}}^{-1} =$ the Azimuthal Spherical CS inverse generating function
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) The generating function is the composition of the generating function for Azimuthal Spherical CS (see Table 5.12) with the 3D localization operator (see 5.3.6.2). 2) The commonly used coordinate-component orderings are either (ρ,α,θ) or (α,θ,ρ). The coordinate-component ordering has been specified as (α,ρ,θ) to ensure that this CS is right-handed. Compliant coordinate value representations are delineated in 8.5.1. 3) The inverse generating function is discontinuous on the z-axis.
References	[EDM]

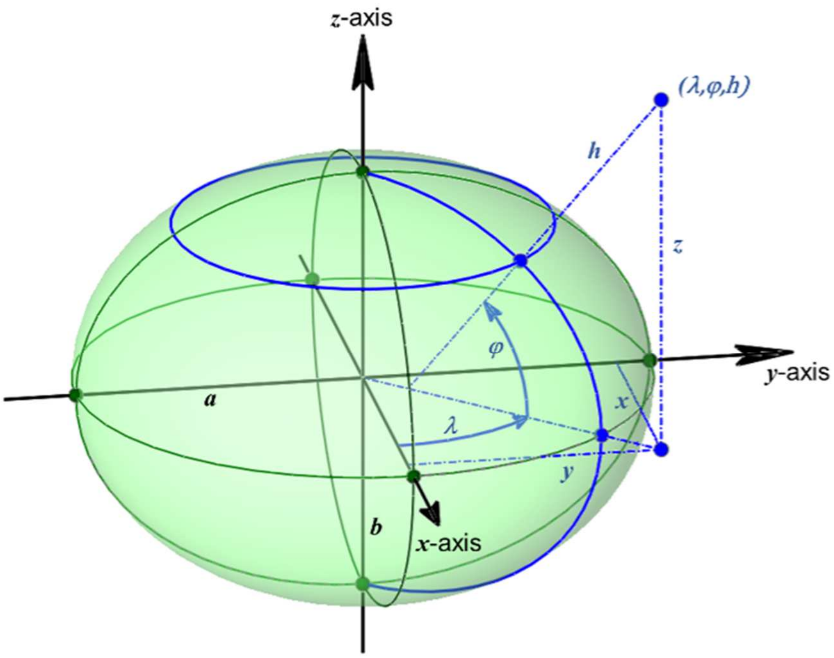
5.3.8.8 Geodetic 3D CS specification

Table 5.14 — Geodetic 3D CS

Element	Specification
Description	Geodetic 3D
CS label	GEODETIC
CS code	7
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal

Element	Specification
CS parameters and constraints	a: major semi-axis length b: minor semi-axis length Constraints: $a > b$: (oblate ellipsoid) $a = b$: (sphere)
Coordinate-components	λ: longitude in radians, φ: geodetic latitude in radians, and h: ellipsoidal height.
CS Domain	$\left\{(\lambda,\varphi,h) \text{ in } \mathbb{R}^3 \,\middle\vert\, -\pi < \lambda \leq \pi \text{ and } \lvert\varphi\rvert < \frac{\pi}{2} \text{ and } -b < h\right\}$ $\cup \left\{(0,\varphi,h) \text{ in } \mathbb{R}^3 \,\middle\vert\, \varphi = \pm\frac{\pi}{2},\ -b < h\right\}$
Generating function	$\boldsymbol{G}\big((\lambda,\varphi,h)\big) = \begin{bmatrix} (\mathcal{N}(\varphi) + h)\cos(\varphi)\cos(\lambda) \\ (\mathcal{N}(\varphi) + h)\cos(\varphi)\sin(\lambda) \\ \big((1-\varepsilon^2)\mathcal{N}(\varphi) + h\big)\sin(\varphi) \end{bmatrix}$ Simplification if $a = b$: $\boldsymbol{G}\big((\lambda,\varphi,h)\big) = \begin{bmatrix} (a + h)\cos(\varphi)\cos(\lambda) \\ (a + h)\cos(\varphi)\sin(\lambda) \\ (a + h)\sin(\varphi) \end{bmatrix}$
Domain of the inverse	$\left\{[x,y,z] \text{ in } \mathbb{R}^3 \,\middle\vert\, (a-b) < \sqrt{x^2+y^2+z^2}\right\} \cup \{[0,0,z] \mid z\ in\ \mathbb{R}\}$

Element	Specification
Inverse	If x=y=0, $\boldsymbol{G}^{-1}([x,y,z]^{\top}) = \begin{cases} \left(0,+\dfrac{\pi}{2},+z-b\right) & z \geq 0 \\ \left(0,-\dfrac{\pi}{2},-z-b\right) & z < 0 \end{cases}$ else $\boldsymbol{G}^{-1}([x,y,z]^{\top}) = (\lambda,\varphi,h)$ where: $\lambda = \text{arctan2}(y,x)$ $\varphi = \text{arctan2}\left(w, z + \varepsilon'^2 \mathrm{z}_0\right)$ $h = u\left(1 - \dfrac{b^2}{av}\right)$ $w = \sqrt{x^2+y^2}$ $\mathrm{z}_0 = \dfrac{b^2 z}{av}$ $u = \sqrt{(w-\varepsilon^2 \mathrm{w}_0)^2 + z^2}$ $v = \sqrt{(w-\varepsilon^2 \mathrm{w}_0)^2 + (1-\varepsilon^2)z^2}$ $\mathrm{w}_0 = \dfrac{-p\varepsilon^2 w}{1+q} + \sqrt{\dfrac{a^2}{2}\left(1+\dfrac{1}{q}\right) - \dfrac{p(1-\varepsilon^2)z^2}{q(1+q)} - \dfrac{p\,w^2}{2}}$ $q = \sqrt{1+2\varepsilon^4 p}$ $p = \dfrac{F}{3(s+1/s+1)^2 G^2}$ $s = \sqrt[3]{1+d+\sqrt{d(d+2)}}$ $d = \dfrac{\varepsilon^4 F w^2}{G^3}$ $G = w^2 + (1-\varepsilon^2)z^2 - \varepsilon^2(a^2-b^2)$ $F = 54b^2z^2$ Simplification if $a = b$: $\varphi = \arcsin(z/(h+a))$ principal value $h = \sqrt{x^2+y^2+z^2} - a$
COM	n/a
Point distortion	n/a

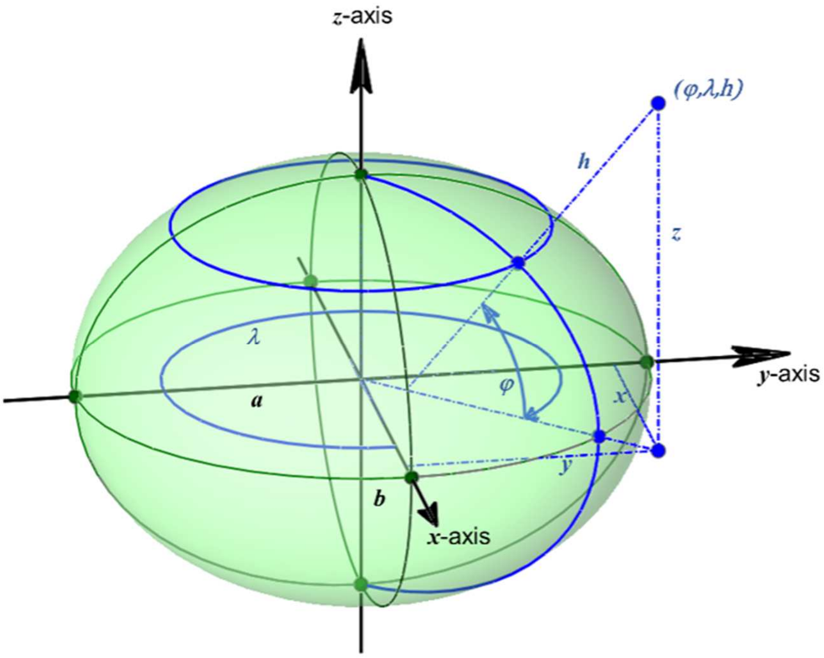
Element	Specification
Figures	
Notes	1) The Surface Geodetic CS (Table 5.24) is the 3rd induced surface CS for this CS at any coordinate for which $h = 0$ (see 5.3.4.2). 2) If $a = b$, the geodetic latitude φ coincides with the spherical latitude θ (see Table 5.10). 3) The inverse generating function is not continuous on the oblate ellipsoid rotational axis. 4) There are various iterative methods to evaluate the inverse generating function such as: $\lambda = \text{atan2}(y, x)$ $\varphi = \lim_{i\to\infty} \varphi^{(i)}$ $h = r\cos\varphi + z\sin\varphi - a\sqrt{1-\varepsilon^2\sin^2\varphi}$ where: $\varphi^{(0)} = \text{atan}\left(r, \frac{z}{1-\varepsilon^2}\right)$ $\varphi^{(i+1)} = \text{atan}\left(r, z + \varepsilon^2 \sin\varphi^{(i)} \mathcal{N}\left(\varphi^{(i)}\right)\right)$ $r = \sqrt{x^2 + y^2}$
References	[HEIK]

5.3.8.9 Planetodetic 3D specification

Table 5.15 — Planetodetic CS

Element	Specification
Description	Planetodetic 3D. Geodetic 3D with longitude in opposite direction
CS label	PLANETODETIC

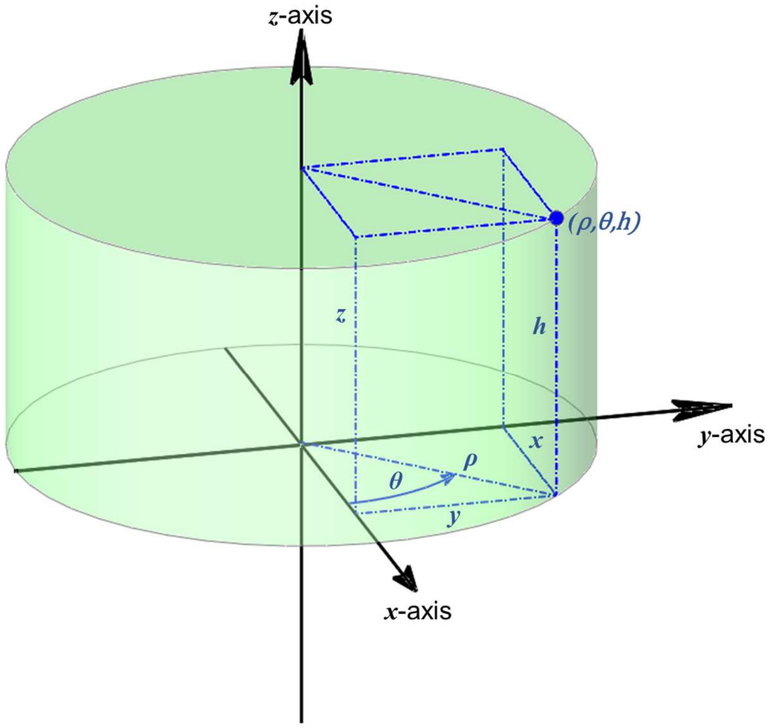
Element	Specification
CS code	8
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	a: major semi-axis length b: minor semi-axis length Constraints: $a > b$: (oblate ellipsoid) $a = b$: (sphere)
Coordinate-components	φ: geodetic latitude in radians, λ: planetodetic longitude in radians, and h: ellipsoidal height.
CS Domain	$\left\{(\varphi, \lambda, h) \text{ in } \mathbb{R}^3 \middle\| -\pi < \lambda \leq \pi \text{ and } \|\varphi\| < \frac{\pi}{2} \text{ and } -b < h\right\}$ $\cup \left\{(0, \lambda\ h) \text{ in } \mathbb{R}^3 \middle\| \lambda = \pm\frac{\pi}{2},\ -b < h\right\}$
Generating function	$\boldsymbol{G}(\varphi, \lambda, h) = \boldsymbol{G}_{\text{GD}}(-\lambda, \varphi, h)$, where $\boldsymbol{G}_{\text{GD}}$ is the Geodetic 3D CS generating function.
Domain of the inverse	$\left\{[x, y, z] \text{ in } \mathbb{R}^3 \middle\| (a - b) < \sqrt{x^2 + y^2 + z^2}\right\} \cup \{[0,0,z] \mid z \text{ in } \mathbb{R}\}$
Inverse	$\boldsymbol{G}^{-1}([x, y, z]^\top) = \boldsymbol{G}_{\text{GD}}^{-1}([x, -y, z]^\top) = (\varphi, \lambda, h)$ where $\boldsymbol{G}_{\text{GD}}^{-1}$ is the Geodetic 3D CS inverse generating function.
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) Similar to the Geodetic 3D CS (see Table 5.14) except that longitude increases in the opposite direction. In particular, points on a planet surface rotating (prograde) into view have larger planetodetic longitudes than those points rotating out of view. This CS is also termed planetocentric when $a = b$ and planetographic when $a > b$. 2) The inverse generating function is not continuous on the oblate ellipsoid rotational axis. 3) The coordinate-component ordering differs from that of Geodetic 3D CS to satisfy the right handedness requirement.
References	[RIIC06]

5.3.8.10 Cylindrical CS specification

Table 5.16 — Cylindrical CS

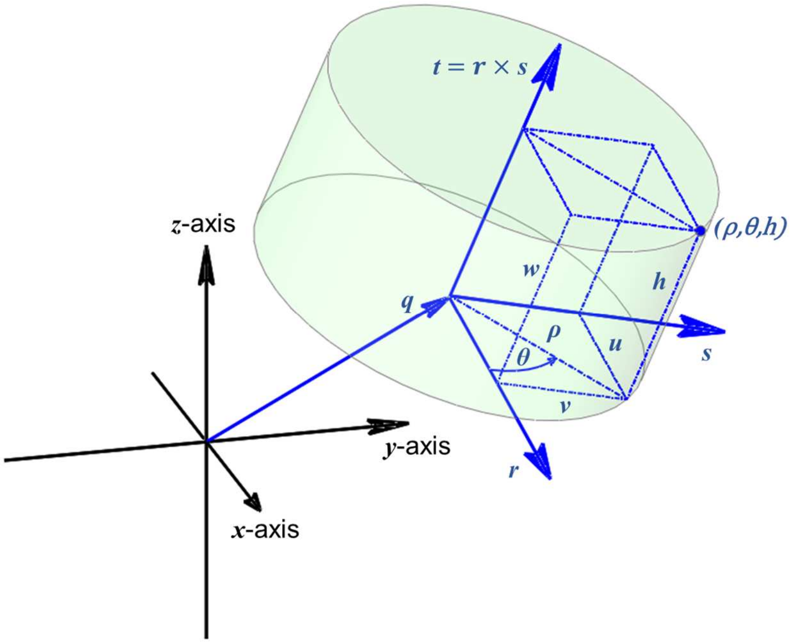
Element	Specification
Description	Cylindrical
CS label	CYLINDRICAL
CS code	9
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	none

Element	Specification
Coordinate-components	ρ: radius, θ: cylindrical angle in radians, and h: height.
CS Domain	$\{(\rho,\theta,h)$ in $\mathbb{R}^3 \mid 0<\rho$ and $0\leq\theta<2\pi\} \cup \{(0,0,h) \mid h$ in $\mathbb{R}\}$
Generating function	$\boldsymbol{G}_{\mathrm{C}}\big((\rho,\theta,h)\big)=\begin{bmatrix}\rho\cos(\theta)\\ \rho\sin(\theta)\\ h\end{bmatrix}$
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}_{\mathrm{C}}^{-1}([x,y,z]^{\top})=(\rho,\theta,h)$, where: $\rho=\sqrt{x^2+y^2}$, $\theta=\begin{cases}\operatorname{arctan2}(y,x) & \text{if } y\geq 0\\ 2\pi+\operatorname{arctan2}(y,x) & \text{if } y<0,\end{cases}$ and $h=z$.
COM	n/a
Point distortion	n/a
Figures	
Notes	The inverse generating function is discontinuous on the half plane $\{[x,0,z]^{\top} \mid 0\leq x\}$
References	[EDM]

5.3.8.11 Lococentric Cylindrical CS specification

Table 5.17 — Lococentric Cylindrical CS

Element	Specification
Description	Localization of the Cylindrical CS
CS label	LOCOCENTRIC_CYLINDRICAL
CS code	10
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	ρ: radius, θ: cylindrical angle in radians, and h: height.
CS Domain	$\{(\rho,\theta,h)$ in $\mathbb{R}^3 \mid 0 < \rho$ and $0 \le \theta < 2\pi\} \cup \{(0,0,h) \mid h$ in $\mathbb{R}\}$
Generating function	$\boldsymbol{G}\big((\rho,\theta,h)\big) = \boldsymbol{L}_{\mathrm{3D}} \circ \boldsymbol{G}_{\mathrm{C}}\big((\rho,\theta,h)\big)$ $= [x,y,z]^{\mathsf{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\ y = \boldsymbol{p} \bullet \boldsymbol{e}_2,\ z = \boldsymbol{p} \bullet \boldsymbol{e}_3$, $\boldsymbol{p} = \boldsymbol{q} + \rho\cos(\theta)\,\boldsymbol{r} + \rho\sin(\theta)\,\boldsymbol{s} + h\boldsymbol{r} \times \boldsymbol{s}$, $\boldsymbol{L}_{\mathrm{3D}} =$ the 3D localization operator, and $\boldsymbol{G}_{\mathrm{C}} =$ the Cylindrical CS generating function
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}^{-1}([x,y,z]^{\mathsf{T}}) = \boldsymbol{G}_{\mathrm{C}}^{-1} \circ \boldsymbol{L}_{\mathrm{3D}}^{-1}([x,y,z]^{\mathsf{T}}) = (\rho,\theta,h)$, where: $\rho = \sqrt{u^2+v^2}$, $\theta = \begin{cases} \mathrm{arctan2}(v,u) & \text{if } v \ge 0 \\ 2\pi + \mathrm{arctan2}(v,u) & \text{if } v < 0, \end{cases}$ $h = w$ $u = \boldsymbol{p} \bullet \boldsymbol{r},\ v = \boldsymbol{p} \bullet \boldsymbol{s},\ w = \boldsymbol{p} \bullet (\boldsymbol{r} \times \boldsymbol{s})$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3) - \boldsymbol{q}$, $\boldsymbol{L}_{\mathrm{3D}}^{-1} =$ the 3D localization inverse operator, and $\boldsymbol{G}_{\mathrm{C}}^{-1} =$ the Cylindrical CS inverse generating function
COM	n/a
Point distortion	n/a

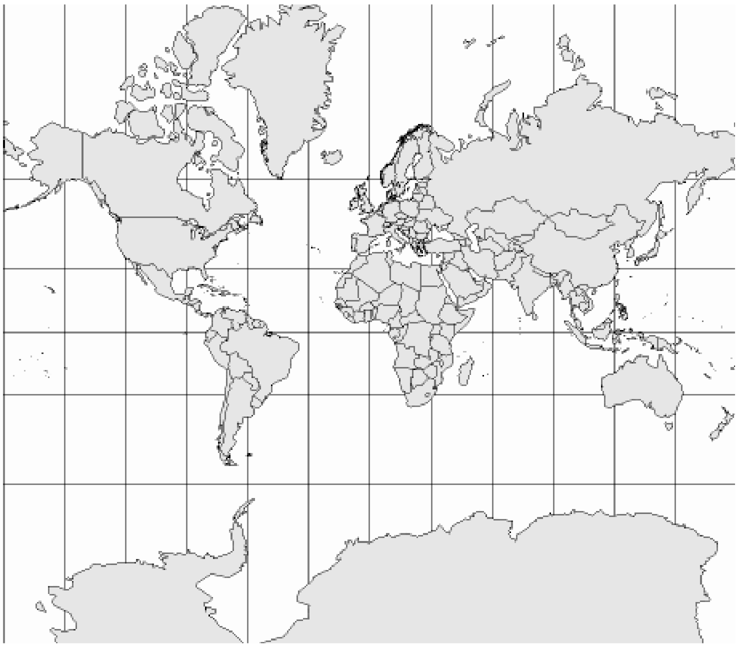
Element	Specification
Figures	
Notes	1) The generating function is the composition of the Cylindrical generating function (see Table 5.16) with the 3D localization operator (see 5.3.6.2). 2) The inverse generating function is discontinuous on the half plane $\{\, [x, y, z]^{\top} \mid ([x, y, z]^{\top} - \boldsymbol{q}) \bullet \boldsymbol{r} \geq 0,\ ([x, y, z]^{\top} - \boldsymbol{q}) \bullet \boldsymbol{s} = 0\}$
References	[EDM]

5.3.8.12 Mercator CS specification

Table 5.18 — Mercator CS

Element	Specification
Description	Mercator and augmented Mercator map projection coordinate systems
CS label	MERCATOR
CS code	11
Function type	Mapping equations
CS descriptor	Surface (map projection) and 3D (augmented map projection)
Properties	Orthogonal, conformal
CS parameters and constraints	a: oblate ellipsoid major semi-axis $(a > 0)$ ε: oblate ellipsoid eccentricity $(0 \leq \varepsilon < 1)$ λ_{origin}: longitude of origin in radians $\left(-\pi < \lambda_{\text{origin}} \leq \pi\right)$ k_0: central scale $(0 < k_0 \leq 1)$ u_F: false easting v_F: false northing

Element	Specification
Coordinate-components	u: easting, and v: northing. Augmented coordinate: h: ellipsoidal height
MP Domain	$\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \middle\vert -\pi < \lambda - \lambda_{\text{origin}} \leq \pi \text{ and } -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right\}$
Mapping equations	$u = P_1(\lambda, \varphi) = u_{\mathrm{F}} + ak_0\Lambda^*$, and $v = P_2(\lambda, \varphi) = v_{\mathrm{F}} + ak_0 \ln\left(\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\left(\frac{1 - \varepsilon\sin(\varphi)}{1 + \varepsilon\sin(\varphi)}\right)^{\frac{\varepsilon}{2}}\right)$, where: $\Lambda^* = \Lambda_{\mathrm{C}}(\lambda, -\lambda_{\text{origin}})$.
Domain of the inverse	$\{[u, v] \text{ in } \mathbb{R}^2 \vert -\pi ak_0 < u - u_{\mathrm{F}} \leq \pi ak_0\}$
Inverse	$\lambda = Q_1(u, v) = \Lambda_C(\Lambda^*, -\lambda_{\text{origin}})$, where: $\Lambda^* = \frac{u - u_{\mathrm{F}}}{ak_0}$ For φ, functional iteration is used for the representation of the inverse mapping equation [SNYD]. Superscripts involving i indicate elements in the iteration sequence. $\varphi = Q_2(u, v) = \lim_{i\to\infty} Q_2^i(u, v)$, $Q_2^1(u, v) = \frac{\pi}{2} - 2\arctan\left(\exp\left(\frac{-v + v_{\mathrm{F}}}{ak_0}\right)\right)$, and $Q_2^{i+1}(u, v) = \frac{\pi}{2} - 2\arctan\left(\exp\left(\frac{-v+v_{\mathrm{F}}}{ak_0}\right)\left(\frac{1-\varepsilon\sin\left(Q_2^i(u,v)\right)}{1+\varepsilon\sin\left(Q_2^i(u,v)\right)}\right)^{\frac{\varepsilon}{2}}\right)$, for $i = 1, 2, 3, \ldots$
COM	$\gamma(\lambda, \varphi) = 0$
Point distortion	$k(\lambda, \varphi) = \frac{ak_0}{\mathcal{N}(\varphi)\cos(\varphi)}$

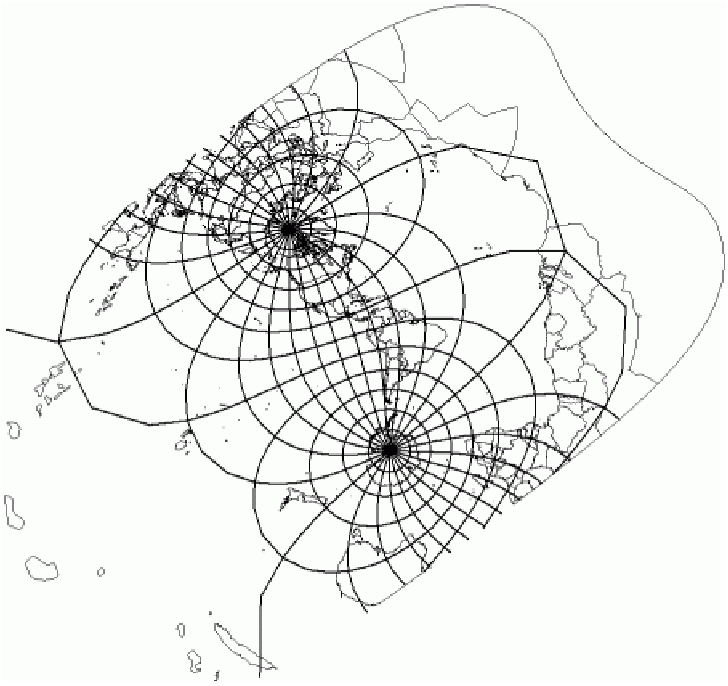
Element	Specification
Figures	 Example Mercator map projection
Notes	1) Meridians project as straight lines that satisfy equations of the form $u =$ some constant. Equally-spaced meridians project to equally-spaced straight lines orthogonal to the u-axis. Parallels project to straight lines orthogonal to the projected meridians and satisfy equations of the form $v =$ some constant. Evenly-spaced parallels project to unevenly-spaced parallel lines on the projection. The spacing of these lines increases with distance from the u-axis. 2) The meridian at λ_{origin} corresponds to the line $u = u_{\text{F}}$. 3) The point distortion equals k_0 along the Equator. 4) An alternate CS parameter set is given by: $a, \varepsilon, \lambda_{\text{origin}}, u_{\text{F}}, u_{\text{F}}$, and φ_1: the secant latitude in radians $(-\pi/2 < \varphi_1 < \pi/2)$. This reduces to the specified CS parameter set by assigning: $k_0 = \frac{1}{a}\,\mathcal{N}(\varphi_1)\cos(\varphi_1)$. With this value for k_0, $k(\lambda, \varphi) = 1$.
References	[SNYD]

5.3.8.13 Oblique Mercator Spherical CS specification

Table 5.19 — Oblique Mercator Spherical CS

Element	Specification
Description	Oblique Mercator and Augmented Oblique Mercator map projections of a sphere
CS label	OBLIQUE_MERCATOR_SPHERICAL
CS code	12
Function type	Mapping equations
CS descriptor	Surface (map projection) and 3D (augmented map projection)

Element	Specification
Properties	Orthogonal, conformal
CS parameters and constraints	R: radius of the sphere $(0 < R)$ k_0: central scale $(0 < k_0 \leq 1)$ (λ_1, φ_1): first point specifying the central line (λ_2, φ_2): second point specifying the central line u_F: false easting v_F: false northing Constraints: $-\frac{\pi}{2} < \varphi_1 \leq \frac{\pi}{2},\quad -\frac{\pi}{2} < \varphi_2 \leq \frac{\pi}{2},\quad \lvert\varphi_1\rvert + \lvert\varphi_2\rvert > 0$, and $-\pi < \lambda_1 \leq \pi,\quad -\pi < \lambda_2 \leq \pi,\quad \lambda_1 \neq \lambda_2,\quad \lvert\lambda_1 - \lambda_2\rvert \neq \pi$.
Coordinate-components	u: easting, and v: northing. Augmented coordinate: h: ellipsoidal height
MP Domain	$\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \middle\vert -\pi < \lambda - \lambda_{\text{origin}} \leq \pi \text{ and } -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\right\}$ except for the transformed pole points determined by the values λ_{origin} and α_0. These values are computed in the forward mapping equations. If $\alpha_0 > 0$, the transformed poles are: $\left(\Lambda_{\text{C}}\left(-\pi/2, -\lambda_{\text{origin}}\right), \ \pi/2 - \alpha_0\right)$ northern hemisphere transformed pole, and $\left(\Lambda_{\text{C}}\left(\pi/2, -\lambda_{\text{origin}}\right), \ -\pi/2 + \alpha_0\right)$ southern hemisphere transfromed pole. If $\alpha_0 < 0$, the transformed poles are: $\left(\Lambda_{\text{C}}\left(\pi/2, -\lambda_{\text{origin}}\right), \ \pi/2 + \alpha_0\right)$ northern hemisphere transformed pole, and $\left(\Lambda_{\text{C}}\left(-\pi/2, -\lambda_{\text{origin}}\right), \ -\pi/2 - \alpha_0\right)$ southern hemisphere transformed pole.
Mapping equations	$u = P_1(\lambda, \varphi) = u_{\text{F}} + Rk_0 \operatorname{arctan2}\left(p_1(\lambda, \varphi), \ \cos(\varphi)\cos\left(\lambda - \lambda_{\text{origin}}\right)\right)$, $v = P_2(\lambda, \varphi) = v_{\text{F}} + \frac{1}{2} Rk_0 \ln\left(\frac{1 - p_2(\lambda, \varphi)}{1 + p_2(\lambda, \varphi)}\right)$, where: $p_1(\lambda, \varphi) = \sin(\alpha_0)\sin(\varphi) + \cos(\alpha_0)\cos(\varphi)\sin\left(\lambda - \lambda_{\text{origin}}\right)$, $p_2(\lambda, \varphi) = -\cos(\alpha_0)\sin(\varphi) + \sin(\alpha_0)\cos(\varphi)\sin\left(\lambda - \lambda_{\text{origin}}\right)$, $\alpha_0 = \begin{cases} \arctan\left(\dfrac{\sin(\varphi_1)}{\cos(\varphi_1)\sin\left(\lambda_1 - \lambda_{\text{origin}}\right)}\right) & \text{if} \left\lvert\sin\left(\lambda_1 - \lambda_{\text{origin}}\right)\right\rvert \geq \left\lvert\sin\left(\lambda_2 - \lambda_{\text{origin}}\right)\right\rvert \\ \arctan\left(\dfrac{\sin(\varphi_2)}{\cos(\varphi_2)\sin\left(\lambda_2 - \lambda_{\text{origin}}\right)}\right) & \text{if} \left\lvert\sin\left(\lambda_1 - \lambda_{\text{origin}}\right)\right\rvert < \left\lvert\sin\left(\lambda_2 - \lambda_{\text{origin}}\right)\right\rvert \end{cases}$, α_0 is the principal value of the arctangent, $\lambda_{\text{origin}} = \operatorname{arctan2}(p_0, q_0)$, $p_0 = \cos(\varphi_1)\sin(\varphi_2)\sin(\lambda_1) - \sin(\varphi_1)\cos(\varphi_2)\cos(\lambda_2)$, and $q_0 = \cos(\varphi_1)\sin(\varphi_2)\cos(\lambda_1) - \sin(\varphi_1)\cos(\varphi_2)\cos(\lambda_2)$. Note: reversing the points (λ_1, φ_1) and (λ_2, φ_2) will result in the antipodal λ_{origin}.
Domain of the inverse	$\{[u, v] \text{ in } \mathbb{R}^2 \vert -\pi a k_0 < u - u_{\text{F}} < \pi a k_0\}$

Element	Specification
Inverse	$\lambda = Q_1(u,v) = \Lambda_C(\Lambda^*,\ -\lambda_{\text{origin}})$ $\varphi = Q_2(u,v) = \arcsin\left(\dfrac{q_2(u^*,v^*)}{\cosh(v^*)}\right)$ (principal value) where: $\Lambda^* = \text{arctan2}\big(q_1(u^*,v^*), \cos(u^*)\big)$ $q_1(u^*,v^*) = \cos(\alpha_0)\sin(u^*) - \sin(\alpha_0)\sinh(v^*)$, $q_2(u^*,v^*) = \sin(\alpha_0)\sin(u^*) + \cos(\alpha_0)\sinh(v^*)$, $u^* = \dfrac{u-u_F}{Rk_0}$, and $v^* = \dfrac{v-v_F}{Rk_0}$.
COM	$\gamma(\lambda,\varphi) = \text{arctan2}\big(-\sin(\alpha_0)\cos(\lambda-\lambda_{\text{origin}}), \cos(\alpha_0)\cos(\varphi) + \sin(\alpha_0)\sin(\varphi)\sin(\lambda-\lambda_{\text{origin}})\big)$
Point distortion	$k(\lambda,\varphi) = \dfrac{k_0}{\sqrt{1-\big(\cos(\alpha_0)\sin(\varphi)-\sin(\alpha_0)\cos(\varphi)\sin(\lambda-\lambda_{\text{origin}})\big)^2}}$
Figures	 Example Oblique Mercator Spherical map projection

Element	Specification
Notes	1) The method for specifying the central line by specifying two points on the central line can be accomplished by alternative formulations. The formulations for two such alternatives are provided below. Alternative a): In this alternative the user specifies λ_{origin} the longitude of one of the two points where the central line crosses the equator and α_0 the equator crossing angle at that point. The CS parameters are: $\lambda_{\text{origin}}, \alpha_0, k_0, u_F,$ and v_F. The CS parameter constraints are: $0 < \lvert\alpha_0\rvert < \pi/2$, and $-\pi < \lambda_{\text{origin}} \leq \pi$ Alternative b): In this alternative, a point (λ_1, φ_1) on the central line, a central line crossing angle α_1 at the point, and k_0 at the point are specified. The central line crossing angle is the angle between the central line and the parallel through the given point. The positive sense of the angle is counterclockwise from east. In this case the origin $(\lambda_{\text{origin}}, 0)$and the equator crossing α_0 are computed. The CS parameters are: $\lambda_1, \varphi_1, \alpha_1, k_0, u_F,$ and v_F. The CS parameter constraints are: $0 < \lvert\alpha_1\rvert < \pi/2, -\pi < \lambda_1 \leq \pi$, $\lvert\varphi_1\rvert \neq \pi/2$ The mathematical formulation is: $\Lambda_1^* = \text{arctan2}(\cos(\alpha_1)\sin(\varphi_1), \sin(\alpha_1))$, $\lambda_{\text{origin}} = \Lambda_C(\lambda_1, \Lambda_1^*)$, and $\alpha_0 = \arccos(\cos(\alpha_1)\cos(\varphi_1))$ (principal value). 2) Point distortion is equal to k_0 at all points on the central line. 3) The longitude of origin λ_{origin} is related to the pole longitude λ_p in [SNYD] by $\lambda_{\text{origin}} = \lambda_p + \pi/2$.
References	[SNYD], [THOM]

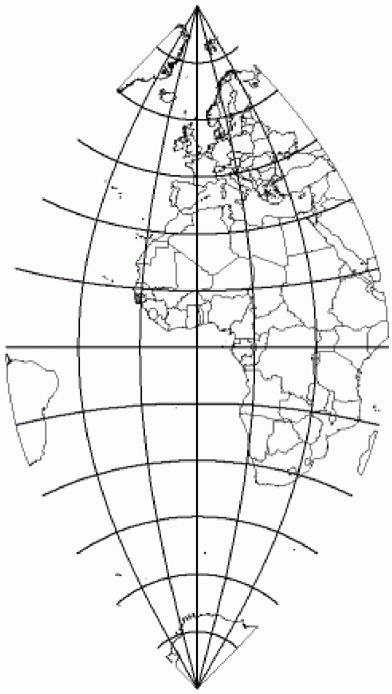
5.3.8.14 Transverse Mercator CS specification

Table 5.20 — Transverse Mercator CS

Element	Specification
Description	Transverse Mercator and Augmented Transverse Mercator map projections
CS label	TRANSVERSE_MERCATOR
CS code	13
Function type	Mapping equations
CS descriptor	Surface (map projection) and 3D (augmented map projection)
Properties	Orthogonal, conformal
CS parameters and constraints	a: oblate ellipsoid major semi-axis $(a > 0)$ ε: oblate ellipsoid eccentricity $(0 \leq \varepsilon < 1)$ λ_{origin}: longitude of origin in radians $(-\pi < \lambda_{\text{origin}} \leq \pi)$ φ_{origin}: latitude of origin in radians $(-\pi < \varphi_{\text{origin}} \leq \pi)$ k_0: central scale $(0 < k_0)$ u_F: false easting v_F: false northing

Element	Specification
Coordinate-components	u: easting, and v: northing. Augmented coordinate: h: ellipsoidal height
MP Domain	$\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \middle\| -\pi < \lambda - \lambda_{\text{origin}} \le \pi \text{ and } -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \text{ and not } \mathbf{A}\right\}$ Condition **A**: $\varphi = 0$ and $(1-\varepsilon)\frac{\pi}{2} \le \left\|\Lambda_{\text{C}}(\lambda, \lambda_{\text{origin}})\right\| \le (1+\varepsilon)\frac{\pi}{2}$
Mapping equations	$u = P_1(\lambda, \varphi) = u_{\text{F}} + ak_o u^*$, and $v = P_2(\lambda, \varphi) = v_{\text{F}} + k_o\left(av^* - S(\varphi_{\text{origin}})\right)$, where: $v^* + iu^* = (1-\varepsilon^2)\int_0^w \frac{dt}{[dn(t/\varepsilon^2)]^2}, i^2 = -1$ w is the solution to the equation: $f(w) = \psi + i\Lambda^*$, with $\psi = \text{arctanh}\left(\sin(\varphi)\right) - \varepsilon\,\text{arctanh}\left(\varepsilon \cdot \sin(\varphi)\right)$, $\Lambda^* = \Lambda_{\text{C}}(\lambda,\ \lambda_{\text{origin}})$, and $f(w) = \text{arctanh}\left(\text{sn}(w\|\varepsilon^2)\right) - \varepsilon \cdot \text{arctanh}\left(\varepsilon \cdot \text{sn}(w\|\varepsilon^2)\right)$ The solution w is determined by Newton's method for complex functions, $w = \lim_{m \to \infty} w^m$ $w^0 = \arcsin(\tanh(\psi + i\Lambda^*))$ (principal value), $w^{m+1} = w^m - \frac{f(w^m) - (\psi + i\Lambda^*)}{f'(w^m)}, \quad m = 0,1,2,3\ldots$, and $f'(w) = \frac{1-\varepsilon^2}{\text{cn}(w\|\varepsilon^2) \cdot \text{dn}(w\|\varepsilon^2)}$.
Domain of the inverse	$\left\{[u, v] \text{ in } \mathbb{R}^2 \middle\| \ \|u - u_{\text{F}}\| < L \text{ and } \|v - v_{\text{F}}\| < 2k_0\mathcal{S}\left(\frac{\pi}{2}\right)\right\}$ where: $L = k_0 a(1-\varepsilon^2)\int_0^{\frac{\pi}{2}} \frac{\sin^2(\xi)}{\sqrt{1-(1-\varepsilon^2)\sin^2(\xi)}}d\xi$

Element	Specification
Inverse	$\lambda = Q_1(u,v) = \Lambda_C(\Lambda^*, -\lambda_{\text{origin}})$, $\varphi = Q_2(u,v)$ where: $\psi + i\Lambda^* = \text{arctanh}(\text{sn}(w\|\varepsilon^2)) - \varepsilon \cdot \text{arctanh}(\varepsilon \cdot \text{sn}(w\|\varepsilon^2)), \quad i^2 = -1.$ w is determined by Newton's method for complex functions: $w = \lim_{m\to\infty} w_m$ $w_0 = v^* + iu^*$, $u^* = \frac{u - u_F}{ak_0}$, $v^* = \frac{v - v_F + k_0 \mathcal{S}(\varphi_{\text{origin}})}{ak_0}$, $w_{m+1} = w_m - \frac{f(w_m) - w_0}{f'(w_m)}, \quad m = 0,1,2,3, \ \dots \ ,$ $f(w) = (1-\varepsilon^2)\int_0^w \frac{dt}{[\text{dn}(t\|\varepsilon^2)]^2}$, $f'(w) = \frac{(1-\varepsilon^2)}{[\text{dn}(w\|\varepsilon^2)]^2}$. φ is determined by functional iteration on the equation for isometric latitude ψ: $\varphi = \lim_{m\to\infty} \varphi_m$, $\varphi_0 = 2\arctan(\exp(\psi)) - \frac{\pi}{2}$, and $\varphi_{m+1} = 2\arctan\left[(\exp(\psi))\left(\frac{1+\varepsilon\sin(\varphi_m)}{1-\varepsilon\sin(\varphi_m)}\right)^{\frac{\varepsilon}{2}}\right] - \frac{\pi}{2}, \qquad m = 0,1,2,3, \ \dots \ .$
COM	$\gamma(\lambda,\varphi) = \text{arctan2}\left(\frac{\partial v^*}{\partial \lambda}, \frac{\partial u^*}{\partial \lambda}\right)$ where: $\frac{\partial v^*}{\partial \lambda} + i\frac{\partial u^*}{\partial \lambda} = i\frac{\text{cn}(w\|\varepsilon^2)}{\text{dn}(w\|\varepsilon^2)}$, and w is the intermediate value computed in the mapping equations for (λ, φ)
Point distortion	$k(\lambda,\varphi) = \frac{ak_o}{\mathcal{N}(\varphi)\,cos(\varphi)}\left\|\frac{\text{cn}(w\|\varepsilon^2)}{\text{dn}(w\|\varepsilon^2)}\right\|$ where w is the intermediate value computed in the mapping equations for (λ, φ) Simplification in the case of a sphere: $k(\lambda,\varphi) = \frac{k_o}{\sqrt{1-\cos^2(\varphi)\sin^2(\lambda - \lambda_{\text{origin}})}}$.

Element	Specification
Figures	 Example Transverse Mercator map projection
Notes	1) As noted in [SNYD] and [LLEE], an iterative exact solution for the transverse Mercator forward and inverse conversions were developed by Prof. E. H. Thompson in 1945 and formally published by L. P. Lee in 1962 [LLEE] with the permission of Professor Thompson. In contrast to approximate forms over limited areas near the central meridian, the Lee/Thompson formulation provides an exact solution over almost all the ellipsoid. The forward and inverse mapping equations used in this International Standard are an adaptation of these based on additional work by C. Rollins of the United States National Geospatial-Intelligence Agency (NGA) to include a central scale factor, a non-zero latitude origin and false easting and false northing offsets for both the easting and northing coordinate-components. 2) The complex functions sn(w |ε^2), cn(w |ε^2) and dn(w |ε^2) are defined in A.8.2. 3) The CS generating function is extended by continuity to include the oblate ellipsoid pole points. 4) The domain of the inverse mapping equations covers the main region of interest. This domain can be extended to a larger region whose exact specification is complicated to define. 5) The iterative procedures used in both the forward and inverse formulations may be numerically ill conditioned near certain points. This occurs near the boundaries of the domains involved and near the equator for points with negative u values. When implementing these methods in software, special numerical methods may be required in the neighbourhood of such exceptional points. In particular, in the forward conversion, the exceptional points on the equator are avoided by restricting the procedure to use |Λ^*| and then setting u to be negative when $\Lambda^* < 0$. 6) The point distortion equals k_0 along the meridian at λ_{origin}.

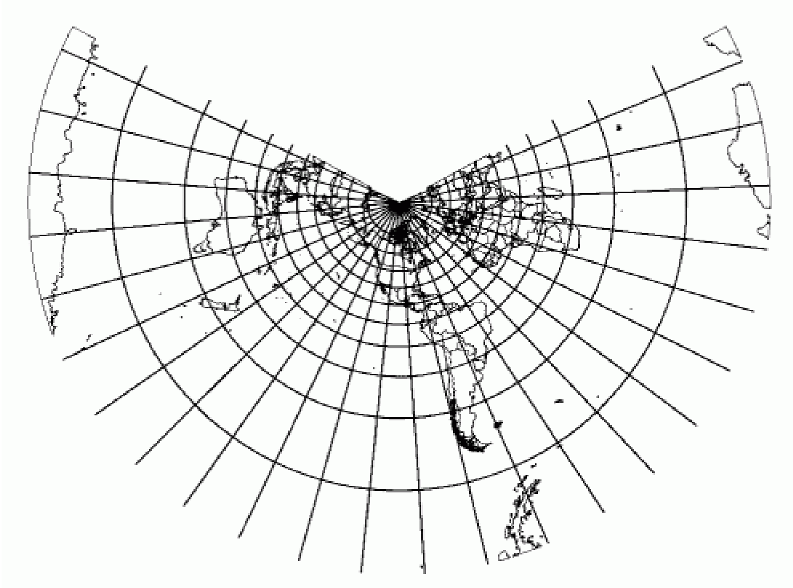
Element	Specification
References	[LLEE], [SNYD], and [DOZI].

5.3.8.15 Lambert Conformal Conic CS specification

Table 5.21 — Lambert conformal Conic CS

Element	Specification
Description	Lambert Conformal Conic and Augmented Lambert Conformal Conic map projections
CS label	LAMBERT_CONFORMAL_CONIC
CS code	14
Function type	Mapping equations
CS descriptor	Surface (map projection) and 3D (augmented map projection)
Properties	Orthogonal, conformal
CS parameters and constraints	a: oblate ellipsoid major semi-axis $(a > 0)$ ε: oblate ellipsoid eccentricity $(0 \leq \varepsilon < 1)$ φ_{origin}: latitude of the origin in radians $(-\pi/2 < \varphi_{\text{origin}} < \pi/2)$ λ_{origin}: longitude of origin in radians $(-\pi < \lambda_{\text{origin}} \leq \pi)$ φ_1, φ_2: secant latitudes in radians $(-\pi/2 < \varphi_1 < \pi/2, -\pi/2 < \varphi_2 < \pi/2)$ φ_1, φ_2: $\varphi_1 \neq -\varphi_2$ u_{F}: false easting v_{F}: false northing
Coordinate-components	u: easting, and v: northing. Augmented coordinate: h: ellipsoidal height
MP Domain	$\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \middle\vert -\pi < \lambda - \lambda_{\text{origin}} \leq \pi \text{ and } -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right\}$

Element	Specification
Mapping equations	$u = P_1(\lambda,\varphi) = u_\mathrm{F} + \rho(\varphi)\sin(n\Lambda^*)$, and $v = P_2(\lambda,\varphi) = v_\mathrm{F} + \rho(\varphi_\text{origin}) - \rho(\varphi)\cos(n\Lambda^*)$, where: $\Lambda^* = \Lambda_\mathrm{C}(\lambda,\ -\lambda_\text{origin})$, $\rho(\varphi) = \rho_0\left(\frac{\tau(\varphi)}{\tau(\varphi_0)}\right)^n$, $\rho_0 = ak_0\frac{m(\varphi_0)}{n}$, $\tau(\varphi) = \tan\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\left[\frac{1+\varepsilon\sin(\varphi)}{1-\varepsilon\sin(\varphi)}\right]^{\frac{\varepsilon}{2}}$, $m(\varphi) = \frac{\cos(\varphi)}{\sqrt{1-\varepsilon^2\sin^2(\varphi)}}$, $k_0 = \frac{m(\varphi_1)}{m(\varphi_0)}\left(\frac{\tau(\varphi_0)}{\tau(\varphi_1)}\right)^n$, $\varphi_0 = \arcsin(n)$ (principal value), and $n = \begin{cases} \sin(\varphi_1) & \text{if } \varphi_1 = \varphi_2 \\ \frac{\ln(m(\varphi_1)) - \ln(m(\varphi_2))}{\ln(\tau(\varphi_1)) - \ln(\tau(\varphi_2))} & \text{if } \varphi_1 \neq \varphi_2 \end{cases}$.
Domain of the inverse	$\left\{ [u,v] \text{ in } \mathbb{R}^2 \,\middle\|\, \begin{matrix} [u,v] \neq (u_\mathrm{F}, v_\mathrm{F} + \rho(\varphi_\text{origin})) \text{ and} \\ \lvert\operatorname{arctan2}(u - u_\mathrm{F}, \rho(\varphi_\text{origin}) - v + v_\mathrm{F})\rvert < \pi\lvert n\rvert \end{matrix} \right\}$
Inverse	$\lambda = Q_1(u,v) = \Lambda_\mathrm{C}(\Lambda^*,\ -\lambda_\text{origin})$, and $\varphi = Q_2(u,v) = \lim_{m\to\infty} Q_2^m(u,v)$, where: $\Lambda^* = \frac{1}{n}\operatorname{arctan2}\left(\operatorname{sgn}(n)(u-u_\mathrm{F}), \operatorname{sgn}(n)(\rho(\varphi_\text{origin}) - v + v_\mathrm{F})\right)$ $Q_2^0(u,v) = \frac{\pi}{2} - 2\arctan(t(u,v))$, $Q_2^{m+1}(u,v) = \frac{\pi}{2} - 2\arctan\left\{t(u,v)\left(\frac{1-\varepsilon\sin(Q_2^m(u,v))}{1+\varepsilon\sin(Q_2^m(u,v))}\right)^{\frac{\varepsilon}{2}}\right\}$, for $m = 1,2,3,\ldots$, $t(u,v) = \tau(\varphi_0)\left(\frac{r(u,v)}{\rho_0}\right)^{\frac{1}{n}}$, $r(u,v) = \operatorname{sgn}(n)\sqrt{(u-u_\mathrm{F}) + [\rho(\varphi_\text{origin}) - v + v_\mathrm{F}]}$, and $n, \rho_0, \tau(\varphi_0)$, and $\rho(\varphi_\text{origin})$ are defined in the forward mapping equations field.
COM	$\gamma(\lambda,\varphi) = n\Lambda^*$ where: $\Lambda^* = \Lambda_C(\lambda,\ \lambda_\text{origin})$.

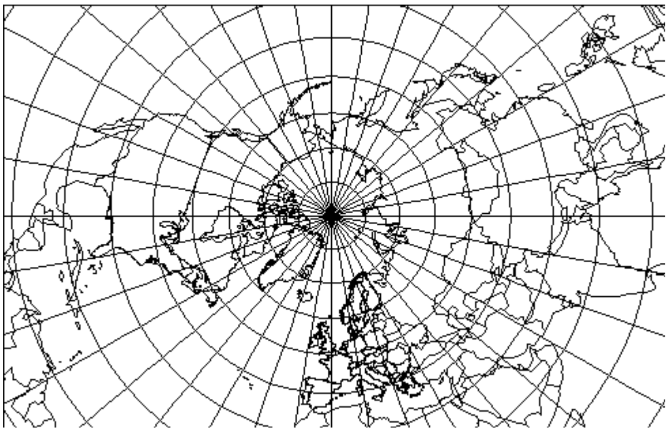
Element	Specification
Point distortion	$k(\lambda, \varphi) = \dfrac{m(\varphi_1)}{m(\varphi)}\left(\dfrac{\tau(\varphi)}{\tau(\varphi_1)}\right)^n = \dfrac{n\rho(\varphi)}{am(\varphi)}$ (equivalent expression)
Figures	 Example Lambert Conformal Conic map projection
Notes	1) $\mathrm{sgn}(x)$ is the signum function (see ISO 80000-2). 2) The Surface Geodetic coordinate $(\lambda_{\text{origin}}, \varphi_{\text{origin}})$projects to map coordinate $(u_{\text{F}}, v_{\text{F}})$. 3) The point distortion is unity along the standard parallel(s) φ_1 and φ_2. 4) An alternate CS parameter set is given by: $a, \varepsilon, \varphi_{\text{origin}}, \lambda_{\text{origin}}, u_{\text{F}}, v_{\text{F}}$, and central scale k_0, $(0 < k_0 \leq 1)$, where k_0 is a user specified point distortion at $(\lambda_{\text{origin}}, \varphi_{\text{origin}})$. In this case, the k_0 replaces the computed intermediate variable in the mapping equations and the computed intermediate variables φ_0 and n are set as $\varphi_0 = \varphi_{\text{origin}}$, and $n = \sin(\varphi_0)$.
References	[SNYD]

5.3.8.16 Polar Stereographic CS specification

Table 5.22 — Polar Stereographic CS

Element	Specification
Description	Polar Stereographic and Augmented Polar Stereographic map projections
CS label	POLAR_STEREOGRAPHIC
CS code	15
Function type	Mapping equations
CS descriptor	Surface (map projection) and 3D (augmented map projection)
Properties	Orthogonal, conformal

Element	Specification
CS parameters and constraints	a: oblate ellipsoid major semi-axis $(a > 0)$ ε: oblate ellipsoid eccentricity $(0 \le \varepsilon < 1)$ polar aspect: north or south λ_{origin}: longitude of origin in radians $(-\pi < \lambda_{\text{origin}} \le \pi)$ $\varphi_{\text{origin}} = \begin{cases} +\pi/2 & \text{north aspect} \\ -\pi/2 & \text{south aspect} \end{cases}$ k_0: central scale $(1/2 \le k_0 \le 1)$ u_{F}: false easting v_{F}: false northing
Coordinate-components	u: easting, and v: northing. Augmented coordinate: h: ellipsoidal height
MP Domain	North aspect: $\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \,\middle|\, -\pi < \lambda - \lambda_{\text{origin}} \le \pi \text{ and } 0 \le \varphi < \frac{\pi}{2}\right\} \cup \left\{\left(0, +\frac{\pi}{2}\right)\right\}$ South aspect: $\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \,\middle|\, -\pi < \lambda - \lambda_{\text{origin}} \le \pi \text{ and } -\frac{\pi}{2} < \varphi \le 0\right\} \cup \left\{\left(0, -\frac{\pi}{2}\right)\right\}$
Mapping equations	North aspect: $u = P_1(\lambda, \varphi) = u_{\mathrm{F}} + \rho(\varphi)\sin(\lambda - \lambda_{\text{origin}})$, $v = P_2(\lambda, \varphi) = v_{\mathrm{F}} - \rho(\varphi)\cos(\lambda - \lambda_{\text{origin}})$ South aspect: $u = P_1(\lambda, \varphi) = u_{\mathrm{F}} + \rho(\varphi)\sin(\lambda - \lambda_{\text{origin}})$, $v = P_2(\lambda, \varphi) = v_{\mathrm{F}} + \rho(\varphi)\cos(\lambda - \lambda_{\text{origin}})$ where: $\rho(\varphi) = 2ak_0E\tau(\varphi)$, $\tau(\varphi) = \tan\left(\frac{\pi}{4} - \frac{\|\varphi\|}{2}\right)\left[\frac{1 + \varepsilon\sin(\|\varphi\|)}{1 - \varepsilon\sin(\|\varphi\|)}\right]^{\frac{\varepsilon}{2}}$, and $E = \frac{b}{a}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{\varepsilon}{2}}$.
Domain of the inverse	$\{[u, v] \text{ in } \mathbb{R}^2 \| \ (u - u_{\mathrm{F}})^2 + (v - v_{\mathrm{F}})^2 < (2ak_0E)^2\}$ E is defined in the forward mapping equations field.

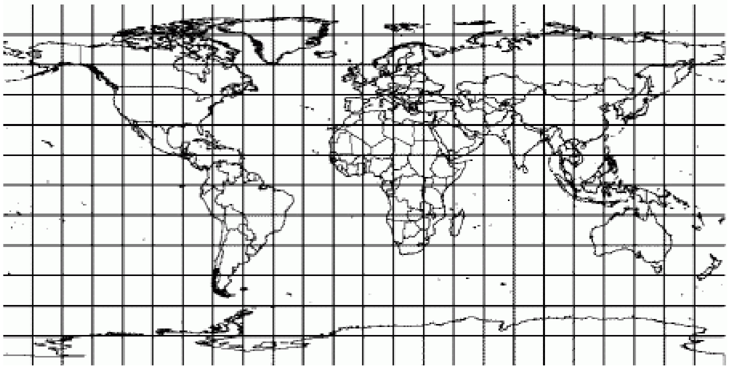
Element	Specification
Inverse	$\lambda = Q_1(u,v) = \Lambda_C(\Lambda^*, -\lambda_{\text{origin}})$, where: $\Lambda^* = \begin{cases} \arctan2(u-u_F, -v+v_F) & \text{north aspect} \\ \arctan2(u-u_F, v-v_F) & \text{south aspect} \end{cases}$ For φ, functional iteration is used for the representation of the inverse mapping equation. Superscripts involving m indicate elements in the iteration sequence. $\varphi = \begin{cases} \lim_{m\to\infty} Q_2^m(u,v) & \text{north aspect} \\ -\lim_{m\to\infty} Q_2^m(u,v) & \text{south aspect} \end{cases}$ where: $Q_2^0(u,v) = \frac{\pi}{2} - 2\arctan\left(t(u,v)\right)$ $Q_2^{m+1}(u,v) = \frac{\pi}{2} - 2\arctan\left\{t(u,v)\left(\frac{1-\varepsilon\sin\left(Q_2^m(u,v)\right)}{1+\varepsilon\sin\left(Q_2^m(u,v)\right)}\right)^{\frac{\varepsilon}{2}}\right\}$ for $m = 1,2,3,\ldots,$ $t(u,v) = \frac{\sqrt{(u-u_F)^2+(v-v_F)^2}}{2ak_0E}$ E is defined in the forward mapping equations field.
COM	$\gamma(\lambda,\varphi) = \begin{cases} \Lambda^* & \text{north aspect} \\ -\Lambda^* & \text{south aspect} \end{cases}$ where: $\Lambda^* = \Lambda_C(\lambda,\ \lambda_{\text{origin}})$.
Point distortion	$k(\lambda,\varphi) = \frac{2ak_0E\tau(\varphi)}{\mathcal{N}(\varphi)\cos(\varphi)}$ where: $\tau(\varphi)$ and E are defined in the forward mapping equations field
Figures	 Example Polar Stereographic map projection See also Figure 5.19.

Element	Specification
Notes	1) Meridians project as straight lines radiating from the point (u_F, v_F). Parallels project to concentric circles. 2) The point distortion values at pole is: $k(\lambda, \varphi_{origin}) = k_0$. 3) An alternate CS parameter set is given by: $a, \varepsilon, \varphi_{\text{origin}}, \lambda_{\text{origin}}, u_F, v_F$, and φ_1: the secant latitude in radians $0 \le \varphi_1 < \pi/2$ north aspect, $-\pi/2 < \varphi_1 \le 0$ south aspect. This reduces to the specified CS parameter set by assigning: $k_0 = \dfrac{\mathcal{N}(\varphi_1)\cos(\varphi_1)}{2aE\tau(\varphi_1)}$ where: $\tau(\varphi)$ and E are defined in the forward mapping equations field, with this value for k_0: $k(\lambda, \varphi_1) = 1$. 4) In the case of a sphere, the mapping equations are derived from the stereographic projection (see Figure 5.19). 5) The CS generating function is extended by continuity to include the oblate ellipsoid pole points: $(\lambda, \varphi) = (0, \pm\pi/2)$.
References	[SNYD]

5.3.8.17 Equidistant Cylindrical CS specification

Table 5.23 — Equidistant Cylindrical CS

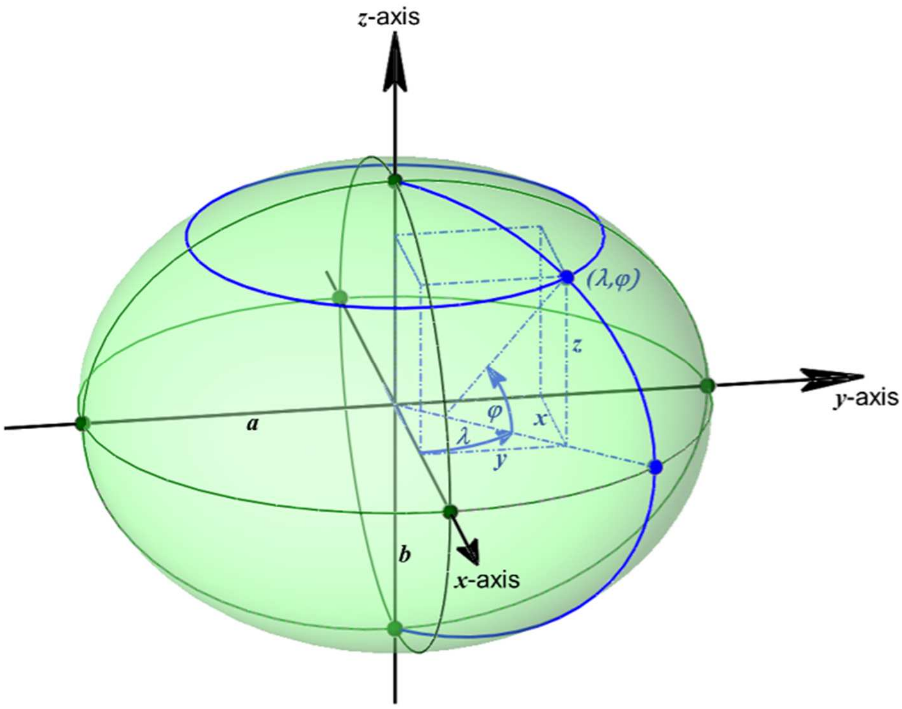
Element	Specification
Description	Equidistant Cylindrical and Augmented Equidistant Cylindrical map projections
CS label	EQUIDISTANT_CYLINDRICAL
CS code	16
Function type	Mapping equations
CS descriptor	Surface (map projection) and 3D (augmented map projection)
Properties	Orthogonal, non-conformal
CS parameters and constraints	a: oblate ellipsoid major semi-axis $(a > 0)$ ε: oblate ellipsoid eccentricity $(0 \le \varepsilon < 1)$ λ_{origin}: longitude of origin in radians $(-\pi < \lambda_{\text{origin}} \le \pi)$ k_0: central scale $(0 < k_0 \le 1)$ u_F: false easting v_F: false northing
Coordinate-components	u: easting, and v: northing. Augmented coordinate: h: ellipsoidal height
MP Domain	$\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \,\middle\|\, -\pi < \lambda - \lambda_{\text{origin}} \le \pi \text{ and } -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right\}$

Element	Specification
Mapping equations	$u = P_1(\lambda, \varphi) = u_{\mathrm{F}} + ak_0\Lambda^*$, $v = P_2(\lambda, \varphi) = v_{\mathrm{F}} + \mathcal{S}(\varphi)$, where: $\Lambda^* = \Lambda_{\mathrm{C}}(\lambda - \lambda_{\text{origin}})$ if $\varepsilon = 0$, $v = P_2(\lambda, \varphi) = u_{\mathrm{F}} + a\varphi$.
Domain of the inverse	$\left\{[u, v] \text{ in } \mathbb{R}^2 \,\middle\vert\, -\pi ak_0 < u - u_{\mathrm{F}} \leq \pi ak_0 \text{ and } -\mathcal{S}\left(\frac{\pi}{2}\right) < v - v_{\mathrm{F}} < \mathcal{S}\left(\frac{\pi}{2}\right)\right\}$
Inverse	$\lambda = Q_1(u, v) = \Lambda_{\mathrm{C}}\left(\frac{u - u_{\mathrm{F}}}{ak_0}, -\lambda_{\text{origin}}\right)$ $\varphi = Q_2(u, v) = \mathcal{S}^{-1}(v - v_{\mathrm{F}})$. if $\varepsilon = 0$, $\varphi = Q_2(u, v) = \frac{v - v_{\mathrm{F}}}{a}$.
COM	$\gamma(\lambda, \varphi) = 0$
Point distortion	$j(\lambda, \varphi) = 1$ latitudinal point distortion $k(\lambda, \varphi) = \frac{ak_0}{\mathcal{N}(\varphi)\cos(\varphi)}$ longitudinal point distortion if $\varepsilon = 0$, $k(\lambda, \varphi) = \frac{k_0}{\cos(\varphi)}$
Figures	 Example Equidistant Cylindrical map projection
Notes	1) Meridians project as straight lines that satisfy equations of the form u = some constant. Equally-spaced meridians project to evenly-spaced straight lines orthogonal to the u-axis. Parallels project to straight lines orthogonal to the projected meridians and satisfy equations of the form v = some constant. 2) $k(\lambda, \varphi) = k_0$ on the equator ($\varphi = 0$). $j(\lambda, \varphi) = 1$ indicates true scale, or "equidistance" along meridians. 3) The radius R of the conceptual cylinder is $R = ak_0$. 4) An alternate CS parameter set is given by: $a, \varepsilon, \lambda_{\text{origin}}, u_{\mathrm{F}}, v_{\mathrm{F}}$, and φ_1: the northern secant latitude in radians. This reduces to the specified CS parameter set by assigning: $k_0 = \frac{1}{a}\mathcal{N}(\varphi_1)\cos(\varphi_1)$ In this case, $k(\lambda, \varphi_1) = 1$ at the secant latitudes $\varphi = \pm\varphi_1$.
References	[SNYD]

5.3.8.18 Surface Geodetic CS specification

Table 5.24 — Surface Geodetic CS

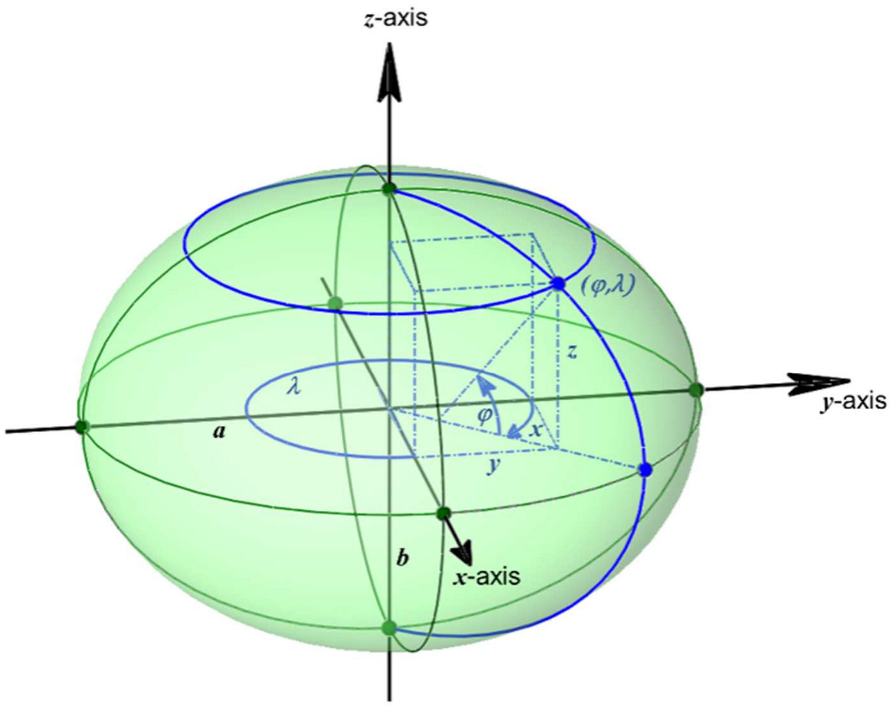
Element	Specification
Description	Surface Geodetic
CS label	SURFACE_GEODETIC
CS code	17
Function type	Generating function
CS descriptor	Surface curvilinear
Properties	Orthogonal
CS parameters and constraints	a: major semi-axis length b: minor semi-axis length Constraints: $a > b$: (oblate ellipsoid) $a = b$: (sphere)
Coordinate-components	λ: longitude in radians, and φ: geodetic latitude in radians.
CS Domain	$\left\{(\lambda, \varphi) \text{ in } \mathbb{R}^2 \,\middle\vert\, -\pi < \lambda \leq \pi \text{ and } -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right\} \cup \left\{\left(0, +\frac{\pi}{2}\right), \left(0, -\frac{\pi}{2}\right)\right\}$
Generating function	$\boldsymbol{G}\big((\lambda, \varphi)\big) = \begin{bmatrix} \mathcal{N}(\varphi)\cos(\varphi)\cos(\lambda) \\ \mathcal{N}(\varphi)\cos(\varphi)\sin(\lambda) \\ (1-\varepsilon^2)\mathcal{N}(\varphi)\sin(\varphi) \end{bmatrix} = \boldsymbol{G}_{\text{G3D}}\big((\lambda, \varphi, 0)\big)$ Simplification if: $a = b = r$: $\boldsymbol{G}\big((\lambda, \varphi)\big) = \begin{bmatrix} r\cos(\varphi)\cos(\lambda) \\ r\cos(\varphi)\sin(\lambda) \\ r\sin(\varphi) \end{bmatrix}$
Domain of the inverse	$\left\{[x, y, z] \text{ in } \mathbb{R}^3 \,\middle\vert\, \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\right\}$
Inverse	$\boldsymbol{G}^{-1}([x, y, z]^{\mathsf{T}}) = \left(\text{arctan2}(y, x),\ \text{arctan2}\left(z, (1-\varepsilon^2)\sqrt{x^2+y^2}\right)\right) = (\lambda, \varphi)$ Simplification if: $a = b = r$: $\varphi = \arcsin(z/r)$ (principal value)
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) The CS range is the oblate ellipsoid (or sphere) surface excluding the pole points. 2) This CS is the 3rd induced surface CS for the Geodetic 3D CS (Table 5.14) at any coordinate for which $h = 0$ (see 5.3.4.2). 3) If $a = b$, the geodetic latitude φ coincides with the spherical latitude θ (see Table 5.10). 4) The inverse generating function is not continuous at the pole points $(0, \pm\pi/2)$.
References	[HEIK]

5.3.8.19 Surface Planetodetic CS specification

Table 5.25 — Surface Planetodetic CS

Element	Specification
Description	Surface Planetodetic. Surface Geodetic with longitude in opposite direction
CS label	SURFACE_PLANETODETIC
CS code	18
Function type	Generating function
CS descriptor	Surface curvilinear
Properties	Orthogonal

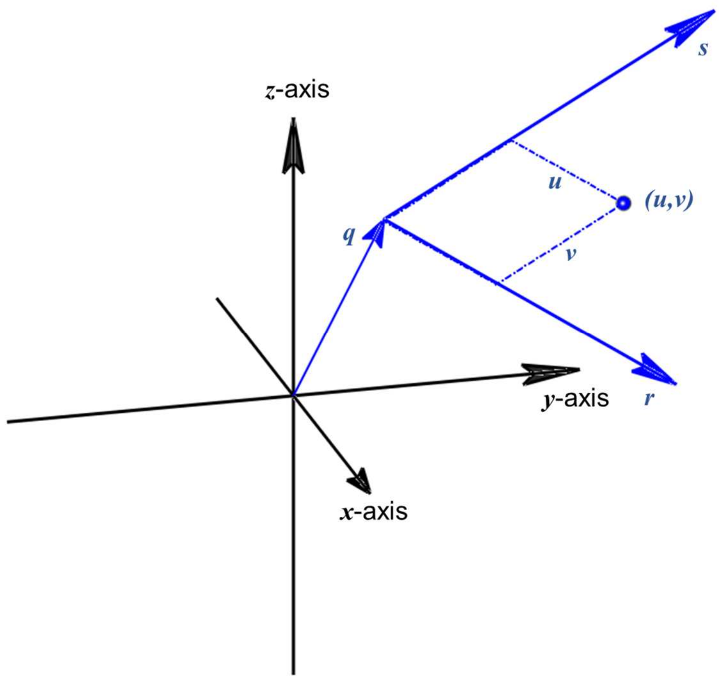
Element	Specification
CS parameters and constraints	a: major semi-axis length b: minor semi-axis length Constraints: $a > b$: (oblate ellipsoid) $a = b$: (sphere)
Coordinate-components	φ geodetic latitude in radians, and λ: planetodetic longitude in radians.
CS Domain	$\left\{(\varphi,\lambda) \text{ in } \mathbb{R}^2 \,\middle\vert\, -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \text{ and } -\pi < \lambda \leq \pi\right\} \cup \left\{\left(0,+\frac{\pi}{2}\right), \left(0,-\frac{\pi}{2}\right)\right\}$
Generating function	$\boldsymbol{G}\big((\varphi,\lambda)\big) = G_{\mathrm{GD}}\big((-\lambda,\varphi)\big)$, where: $\boldsymbol{G}_{\mathrm{GD}}$ is the Surface Geodetic CS generating function
Domain of the inverse	$\left\{[x,y,z] \text{ in } \mathbb{R}^3 \,\middle\vert\, \frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1\right\}$
Inverse	$\boldsymbol{G}^{-1}([x,y,z]^{\mathsf{T}}) = (\varphi,\lambda)$ where: $(\lambda,\varphi) = \boldsymbol{G}_{\mathrm{GD}}^{-1}([x,-y,z]^{\mathsf{T}})$, and $\boldsymbol{G}_{\mathrm{GD}}^{-1}$ is the Surface Geodetic CS inverse generating function.
COM	n/a
Point distortion	n/a
Figures	

Element	Specification
Notes	1) Similar to surface Geodetic CS (see Table 5.24) except that longitude is in the opposite direction. In particular, points on a planet surface rotating (prograde) into view have larger planetodetic longitudes than those points rotating out of view. 2) The inverse generating function is not continuous at the pole points $(0, \pm\pi/2)$. 3) The coordinate-components are ordered for compatibility with Planetodetic 3D CS (see Table 5.15). 4) This CS is the 3rd induced surface CS for the Planetodetic 3D CS at any coordinate for which $h = 0$ (see 5.3.4.2).
References	[RIIC06]

5.3.8.20 Lococentric Surface Euclidean CS specification

Table 5.26 — Lococentric Surface Euclidean CS

Element	Specification
Description	Localization of the Euclidean 2D CS into a plane surface in 3D position-space
CS label	LOCOCENTRIC_SURFACE_EUCLIDEAN
CS code	19
Function type	Generating function
CS descriptor	Surface linear
Properties	Cartesian
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	u, v
CS Domain	$\mathbb{R}^2$
Generating function	$\boldsymbol{G}\big((u,v)\big) = \boldsymbol{L}_{\text{Surface}} \circ \boldsymbol{G}_{\text{E2D}}\big((u,v)\big)$ $= [x, y, z]^{\mathsf{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1$, $y = \boldsymbol{p} \bullet \boldsymbol{e}_2$, $z = \boldsymbol{p} \bullet \boldsymbol{e}_3$, $\boldsymbol{p} = \boldsymbol{q} + u\,\boldsymbol{r} + v\,\boldsymbol{s}$, $\boldsymbol{L}_{\text{Surface}} =$ the surface localization operator, and $\boldsymbol{G}_{\text{E2D}} =$ the Euclidean 2D CS generating function
Domain of the inverse	$\{\boldsymbol{p} = [x, y, z] \text{ in } \mathbb{R}^3 \mid (\boldsymbol{p} - \boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s}) = 0\}$
Inverse	$\boldsymbol{G}^{-1}([x, y, z]^{\mathsf{T}}) = \boldsymbol{G}_{\text{E2D}}^{-1} \circ \boldsymbol{L}_{\text{Surface}}^{-1}([x, y, z]^{\mathsf{T}})$ $= (u, v)$ where: $u = (\boldsymbol{p} - \boldsymbol{q}) \bullet \boldsymbol{r}$, $v = (\boldsymbol{p} - \boldsymbol{q}) \bullet \boldsymbol{s}$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3)$, $\boldsymbol{L}_{\text{Surface}}^{-1} =$ the inverse surface localization operator, and $\boldsymbol{G}_{\text{E2D}}^{-1} =$ the Euclidean 2D CS inverse generating function

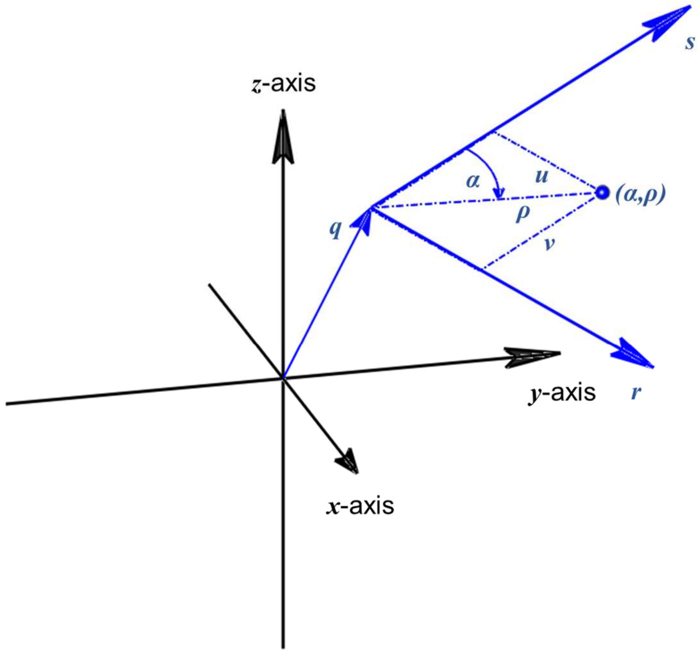
Element	Specification
COM	n/a
Point distortion	n/a
Figures	
Notes	1) The CS range is the plane specified by: $0 = f(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s})$. The generating function is the composition of the generating function for Euclidean 2D (see Table 5.29) with the surface localization operator (see 5.3.6.2). This CS is also the 3rd induced surface CS for the Lococentric Euclidean 3D CS (Table 5.9) at any coordinate for which $w = 0$ (see 5.3.4.2) 2) An alternate CS parameter set is given by: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$: the primary axis direction unit vector in $\mathbb{R}^2$. $\boldsymbol{s}$ is then computed as $\boldsymbol{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{r}$.
References	[EDM]

5.3.8.21 Lococentric Surface Azimuthal CS specification

Table 5.27 — Lococentric Surface Azimuthal CS

Element	Specification
Description	Localization of the Azimuthal CS into a plane surface in 3D position-space.
CS label	LOCOCENTRIC_SURFACE_AZIMUTHAL
CS code	20
Function type	Generating function
CS descriptor	Surface curvilinear

Element	Specification
Properties	Orthogonal
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	α: azimuth in radians, and ρ: radius.
CS Domain	$\{(\alpha,\rho) \text{ in } \mathbb{R}^2 \mid 0 \leq \alpha < 2\pi, \text{ and } 0 < \rho\} \cup \{(0,0)\}$
Generating function	$\boldsymbol{G}\big((\alpha,\rho)\big) = \boldsymbol{L}_{\text{Surface}} \circ \boldsymbol{G}_{\text{A}}\big((\alpha,\rho)\big)$ $= [x,y,z]^{\text{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1$, $y = \boldsymbol{p} \bullet \boldsymbol{e}_2$, $z = \boldsymbol{p} \bullet \boldsymbol{e}_3$, $\boldsymbol{p} = \boldsymbol{q} + \rho(\sin(\alpha)\,\boldsymbol{r} + \cos(\alpha)\,\boldsymbol{s})$, $\boldsymbol{L}_{\text{Surface}} =$ the surface localization operator, and $\boldsymbol{G}_{\text{A}} =$ the Azimuthal CS generating function
Domain of the inverse	$\{\boldsymbol{p} = [x,y,z] \text{ in } \mathbb{R}^3 \mid (\boldsymbol{p}-\boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s}) = 0\}$
Inverse	$\boldsymbol{G}^{-1}([x,y,z]^{\text{T}}) = \boldsymbol{G}_{\text{A}}^{-1} \circ \boldsymbol{L}_{\text{Surface}}^{-1}([x,y,z]^{\text{T}}) = (\alpha,\rho)$, where: $\alpha = \begin{cases} \operatorname{arctan2}(u,v) & \text{if } u \geq 0 \\ 2\pi + \operatorname{arctan2}(u,v) & \text{if } u < 0 \end{cases}$ $\rho = \sqrt{u^2+v^2}$ $u = (\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{r}$, $v = (\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{s}$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3)$, $\boldsymbol{L}_{\text{Surface}}^{-1} =$ the inverse surface localization operator, and $\boldsymbol{G}_{\text{A}}^{-1} =$ the Azimuthal CS inverse generating function
COM	n/a
Point distortion	n/a

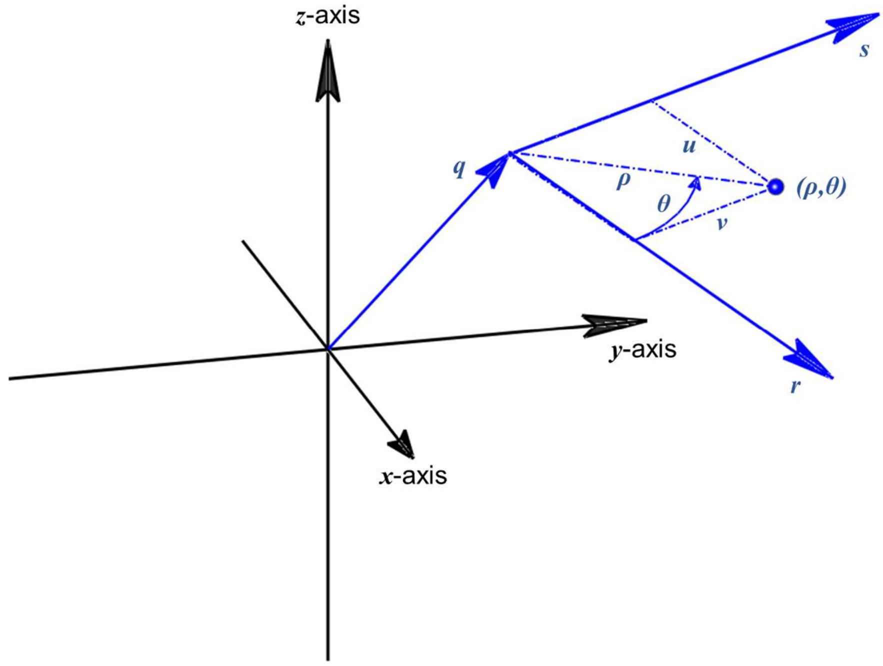
Element	Specification
Figures	
Notes	1) The CS range is the plane specified by: $0 = f(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s})$. 2) The generating function is the composition of the generating function for Azimuthal CS (see Table 5.31) with the surface localization operator (see 5.3.6.2). This CS is also the 3rd induced surface CS for the Lococentric Azimuthal Spherical CS (Table 5.12) at any coordinate for which $\theta = 0$ (see 5.3.4.2). 3) An alternate CS parameter set is given by: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$: the primary axis direction unit vector in $\mathbb{R}^2$. $\boldsymbol{s}$ is then computed as $\boldsymbol{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{r}$.
References	[EDM]

5.3.8.22 Lococentric Surface Polar CS specification

Table 5.28 — Lococentric Surface Polar CS

Element	Specification
Description	Localization of the Polar CS into plane surface in 3D position-space
CS label	LOCOCENTRIC_SURFACE_POLAR
CS code	21
Function type	Generating function
CS descriptor	Surface curvilinear
Properties	Orthogonal

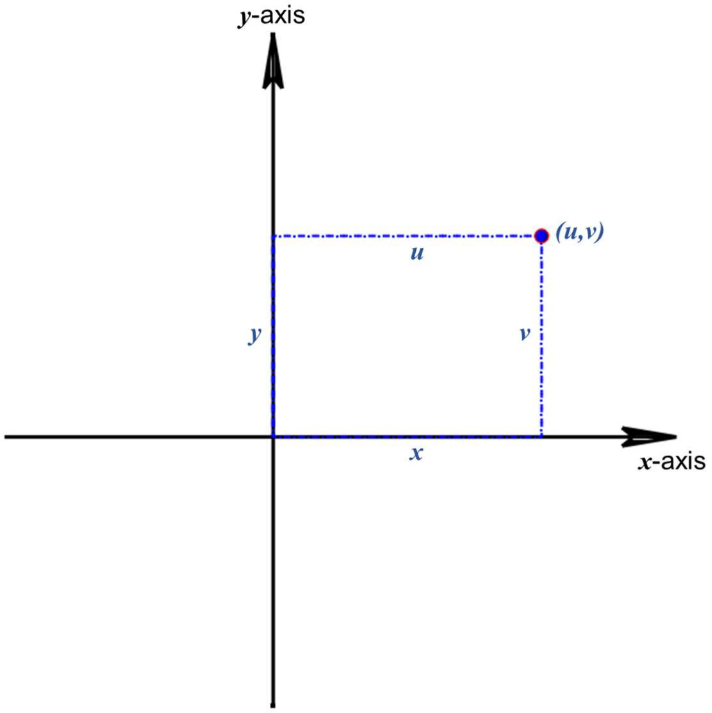
Element	Specification
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	ρ: radius, and θ: angle in radians.
CS Domain	$\{(\rho,\theta) \text{ in } \mathbb{R}^2 \mid 0 < \rho \text{ and } 0 \le \theta < 2\pi\} \cup \{(0,0)\}$
Generating function	$\boldsymbol{G}\big((\rho,\theta)\big) = \boldsymbol{L}_{\text{Surface}} \circ \boldsymbol{G}_{\text{P}}\big((\rho,\theta)\big)$ $= [x,y,z]^{\top}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1$, $y = \boldsymbol{p} \bullet \boldsymbol{e}_2$, $z = \boldsymbol{p} \bullet \boldsymbol{e}_3$, $\boldsymbol{p} = \boldsymbol{q} + \rho\cos(\theta)\ \boldsymbol{r} + \rho\sin(\theta)\ \boldsymbol{s}$, $\boldsymbol{L}_{\text{Surface}} =$ the surface localization operator, and $\boldsymbol{G}_{\text{P}} =$ the Polar CS generating function
Domain of the inverse	$\{\boldsymbol{p} = [x,y,z] \text{ in } \mathbb{R}^3 \mid (\boldsymbol{p}-\boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s}) = 0\}$
Inverse	$\boldsymbol{G}^{-\mathbf{1}}([x,y,z]^{\top}) = \boldsymbol{G}_{\text{P}}^{-1} \circ \boldsymbol{L}_{\text{Surface}}^{-1}\,([x,y,z]^{\top}) = (\rho,\theta)$, where: $\rho = \sqrt{u^2+v^2}$, $\theta = \begin{cases} \arctan2(v,w) & \text{if } v \ge 0 \\ 2\pi + \arctan2(v,w) & \text{if } v < 0, \end{cases}$ $u = (\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{r}$, $v = (\boldsymbol{p}-\boldsymbol{q}) \bullet \boldsymbol{s}$, $w = (\boldsymbol{p}-\boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s})$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3)$, $\boldsymbol{L}_{\text{Surface}}^{-1} =$ the surface localization inverse operator, and $\boldsymbol{G}_{\text{P}}^{-1} =$ the Polar CS inverse generating function
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) The CS range is the plane specified by: $0 = f(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{q}) \bullet (\boldsymbol{r} \times \boldsymbol{s})$. 2) The generating function is the composition of the generating function for Polar CS (see Table 5.33) with the surface localization operator (see 5.3.6.2). This CS is also the 3rd induced surface CS for the Lococentric Cylindrical CS (Table 5.17) at any coordinate for which $h = 0$ (see 5.3.4.2). 3) An alternate CS parameter set is given by: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$: the primary axis direction unit vector in $\mathbb{R}^2$. $\boldsymbol{s}$ is then computed as $\boldsymbol{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{r}$.
References	[EDM]

5.3.8.23 Euclidean 2D CS specification

Table 5.29 — Euclidean 2D CS

Element	Specification
Description	Euclidean 2D
CS label	EUCLIDEAN_2D
CS code	22
Function type	Generating function
CS descriptor	2D linear
Properties	Cartesian

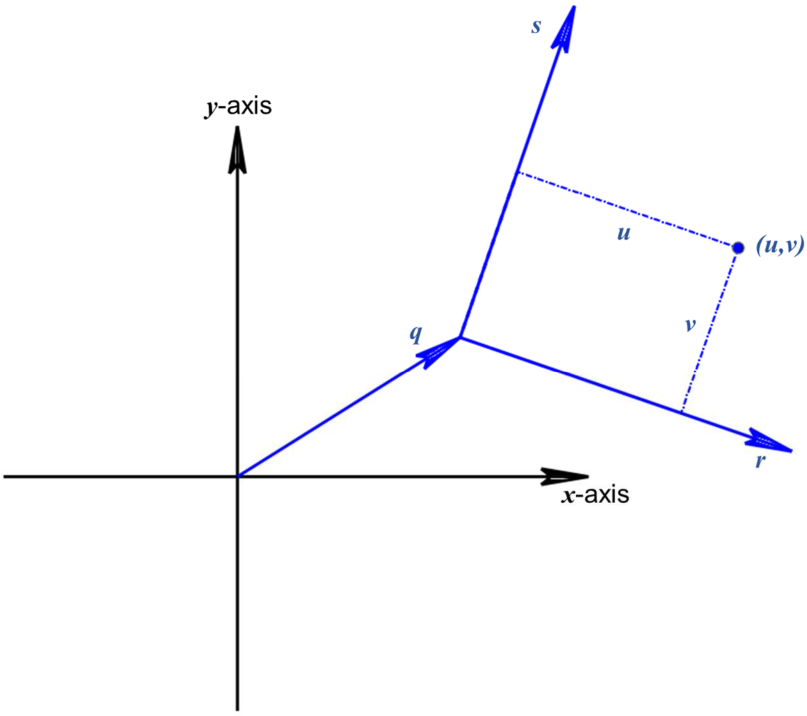
Element	Specification
CS parameters and constraints	none
Coordinate-components	u, v
CS Domain	$\mathbb{R}^2$
Generating function	$\boldsymbol{G}_{\mathrm{E2D}}\big((u,v)\big) = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 = \begin{bmatrix} u \\ v \end{bmatrix}$
Domain of the inverse	$\mathbb{R}^2$
Inverse	$\boldsymbol{G}_{\mathrm{E2D}}^{-1}([u,\ v]^{\mathsf{T}}) = (u,\ v)$
COM	n/a
Point distortion	n/a
Figures	
Notes	Coordinate-space 2-tuples are identified with position-space 2-tuples.
References	[EDM]

5.3.8.24 Lococentric Euclidean 2D CS specification

Table 5.30 — Lococentric Euclidean 2D CS

Element	Specification
Description	Localization of the Euclidean 2D CS.
CS label	LOCOCENTRIC_EUCLIDEAN_2D
CS code	23
Function type	Generating function

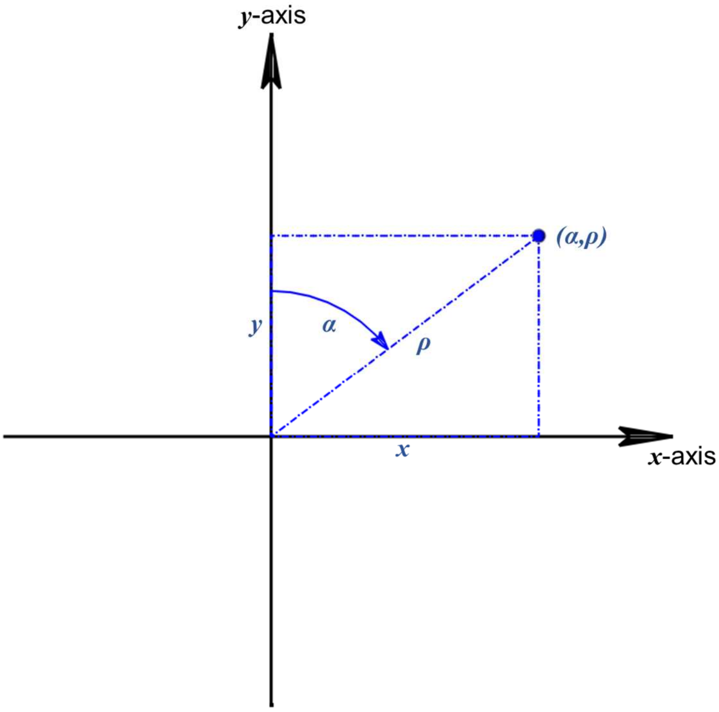
Element	Specification
CS descriptor	2D linear
Properties	Cartesian
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^2$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	u, v
CS Domain	$\mathbb{R}^2$
Generating function	$\boldsymbol{G}\big((u,v)\big) = \boldsymbol{L}_{\text{2D}} \circ \boldsymbol{G}_{\text{E2D}}\big((u,v)\big)$ $= [x,y,z]^{\mathsf{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\ \ y = \boldsymbol{p} \bullet \boldsymbol{e}_2,$ $\boldsymbol{p} = \ \boldsymbol{q} + u\,\boldsymbol{r} + v\,\boldsymbol{s},$ $\boldsymbol{L}_{\text{2D}} =$ the 2D localization operator, and $\boldsymbol{G}_{\text{E2D}} =$ the Euclidean 2D generating function
Domain of the inverse	$\mathbb{R}^2$
Inverse	$\boldsymbol{G}^{-1}\big((u,v)\big) = \boldsymbol{L}_{\text{2D}}^{-1} \circ \boldsymbol{G}_{\text{E2D}}^{-1}\big((u,v)\big)$ $= [x,y,z]^{\mathsf{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\ \ y = \boldsymbol{p} \bullet \boldsymbol{e}_2,$ $\boldsymbol{p} = \ \boldsymbol{q} + u\,\boldsymbol{r} + v\,\boldsymbol{s},$ $\boldsymbol{L}_{\text{2D}}^{-1} =$ the 2D localization operator, and $\boldsymbol{G}_{\text{E2D}}^{-1} =$ the Euclidean 2D generating function
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) Euclidean 2D CS is a special case with $\boldsymbol{q} = (0{,}0)$, $\boldsymbol{r} = (1{,}0)$, $\boldsymbol{s} = (0{,}1)$. 2) The generating function is the composition of the generating function for Euclidean 2D CS (see Table 5.29) with the 2D localization operator (see 5.3.6.2). 3) An alternate CS parameter set is given by: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$: the primary axis direction unit vector in $\mathbb{R}^2$. $\boldsymbol{s}$ is then computed as $\boldsymbol{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{r}$.
References	[EDM]

5.3.8.25 Azimuthal CS specification

Table 5.31 — Azimuthal CS

Element	Specification
Description	Azimuthal coordinate system
CS label	AZIMUTHAL
CS code	24
Function type	Generating function
CS descriptor	2D curvilinear
Properties	Orthogonal
CS parameters and constraints	none

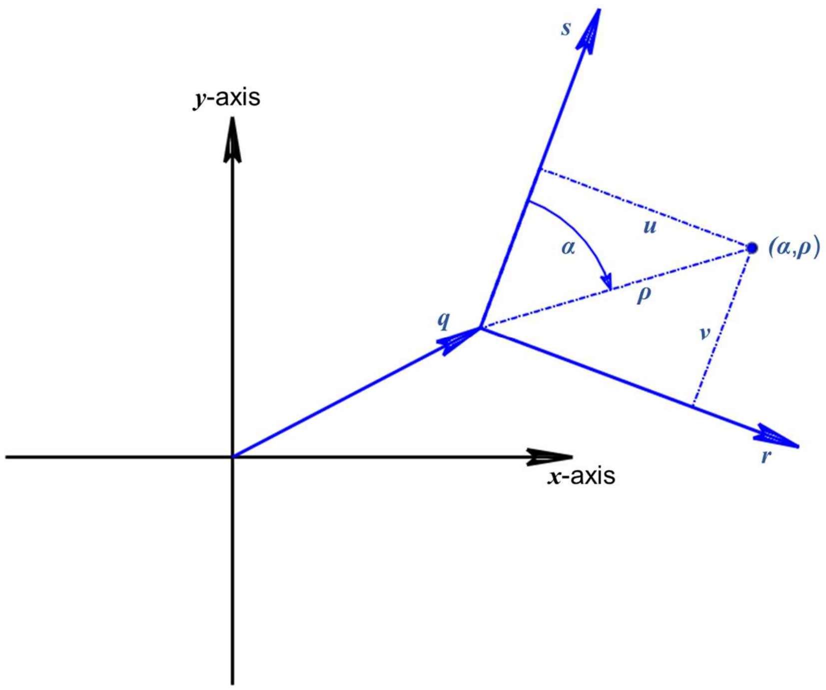
Element	Specification
Coordinate-components	α: azimuth in radians, and ρ: radius.
CS Domain	$\{(\alpha,\rho)$ in $\mathbb{R}^2 \mid 0 \le \alpha < 2\pi$ and $0 < \rho\} \cup \{(0,0)\}$
Generating function	$\boldsymbol{G}_{\mathrm{A}}\big((\alpha,\rho)\big) = \begin{bmatrix}\rho\sin(\alpha)\\ \rho\cos(\alpha)\end{bmatrix}$
Domain of the inverse	$\mathbb{R}^2$
Inverse	$\boldsymbol{G}_{\mathrm{A}}^{-1}([x,y]^{\mathrm{T}}) = (\alpha,\rho)$, where: $\alpha = \begin{cases}\arctan2(x,y) & \text{if } x \ge 0\\ 2\pi + \arctan2(x,y) & \text{if } x < 0\end{cases}$ and $\rho = \sqrt{x^2+y^2}$.
COM	n/a
Point distortion	n/a
Figures	
Notes	The inverse generating function is discontinuous at the CS domain boundary point $(0,0)$.
References	[EDM]

5.3.8.26 Lococentric Azimuthal CS specification

Table 5.32 — Lococentric Azimuthal CS

Element	Specification
Description	Localization of the Azimuthal CS
CS label	LOCOCENTRIC_AZIMUTHAL

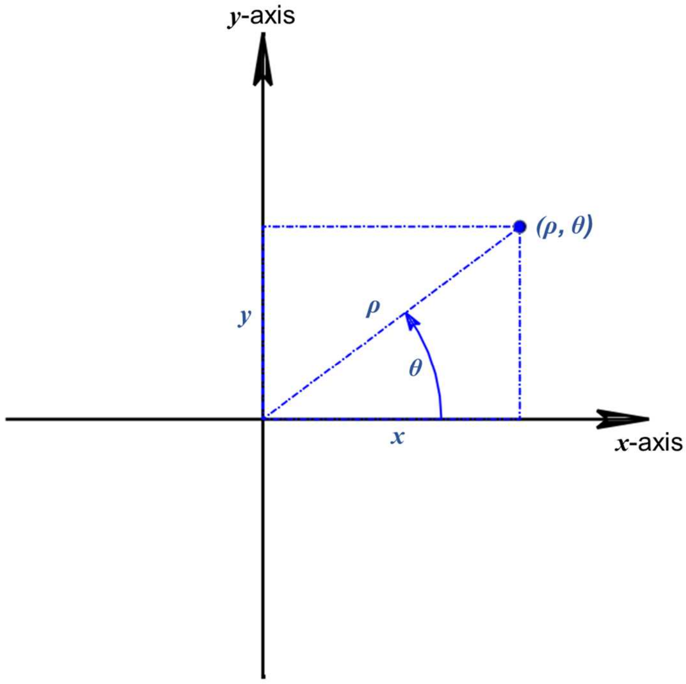
Element	Specification
CS code	25
Function type	Generating function
CS descriptor	2D curvilinear
Properties	Orthogonal
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^2$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	α: azimuth in radians, and ρ: radius.
CS Domain	$\{(\alpha,\rho) \text{ in } \mathbb{R}^2 \mid 0 \leq \alpha < 2\pi \text{ and } 0 < \rho\} \cup \{(0,0)\}$
Generating function	$\boldsymbol{G}\big((\alpha,\rho)\big) = \boldsymbol{L}_{\mathrm{2D}} \circ \boldsymbol{G}_{\mathrm{A}}\big((\alpha,\rho)\big)$ $= [x,y]^{\top}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1$, $y = \boldsymbol{p} \bullet \boldsymbol{e}_2$, $\boldsymbol{p} = \boldsymbol{q} + \rho(\sin(\alpha)\,\boldsymbol{r} + \cos(\alpha)\,\boldsymbol{s})$, $\boldsymbol{L}_{\mathrm{2D}} =$ the 2D localization operator, and $\boldsymbol{G}_{\mathrm{A}} =$ the Azimuthal CS generating function.
Domain of the inverse	$\mathbb{R}^2$
Inverse	$\boldsymbol{G}^{-1}([x,y]^{\top}) = \boldsymbol{G}_{\mathrm{A}}^{-1} \circ \boldsymbol{L}_{\mathrm{2D}}^{-1}([x,y]^{\top}) = (\alpha,\rho)$, where: $\alpha = \begin{cases} \operatorname{arctan2}(u,v) & \text{if } u \geq 0 \\ 2\pi + \operatorname{arctan2}(u,v) & \text{if } u < 0 \end{cases}$ $\rho = \sqrt{u^2+v^2}$ $u = \boldsymbol{p} \bullet \boldsymbol{r}$, $v = \boldsymbol{p} \bullet \boldsymbol{s}$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2) - \boldsymbol{q}$, $\boldsymbol{L}_{\mathrm{2D}}^{-1} =$ the inverse 2D localization operator, and $\boldsymbol{G}_{\mathrm{A}}^{-1} =$ the Azimuthal CS inverse generating function.
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) The generating function is the composition of the generating function for the Azimuthal CS (see Table 5.31) with the 2D localization operator (see 5.3.6.2). 2) An alternate CS parameter set is given by: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$: the primary axis direction unit vector in $\mathbb{R}^2$. $\boldsymbol{s}$ is then computed as $\boldsymbol{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{r}$.
References	[EDM]

5.3.8.27 Polar CS specification

Table 5.33 — Polar CS

Element	Specification
Description	Polar coordinate system
CS label	POLAR
CS code	26
Function type	Generating function
CS descriptor	2D curvilinear
Properties	Orthogonal
CS parameters and constraints	none
Coordinate-components	ρ: radius, and θ: angle in radians.

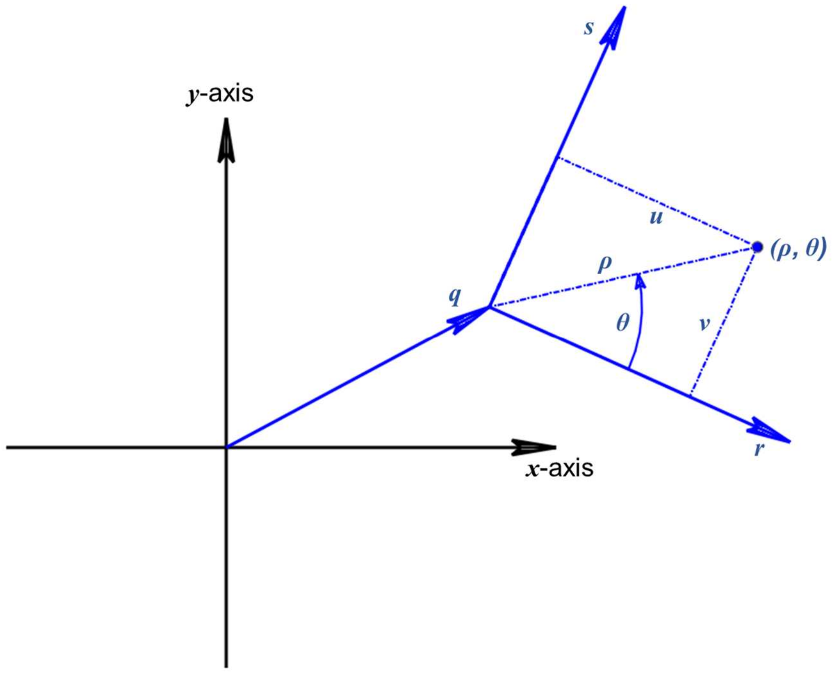
Element	Specification
CS Domain	$\{(\rho,\theta) \text{ in } \mathbb{R}^2 \mid 0 < \rho \text{ and } 0 \le \theta < 2\pi\} \cup \{(0,0)\}$
Generating function	$\boldsymbol{G}_{\mathrm{P}}\big((\rho,\theta)\big) = \begin{bmatrix} \rho\cos(\theta) \\ \rho\sin(\theta) \end{bmatrix}$
Domain of the inverse	$\mathbb{R}^2$
Inverse	$\boldsymbol{G}_{\mathrm{P}}^{-1}([x,y]^{\mathrm{T}}) = (\rho,\theta)$, where: $\theta = \begin{cases} \operatorname{arctan2}(y,x) & \text{if } y \ge 0 \\ 2\pi + \operatorname{arctan2}(y,x) & \text{if } y < 0 \text{, and} \end{cases}$ $\rho = \sqrt{x^2 + y^2}$.
COM	n/a
Point distortion	n/a
Figures	
Notes	The inverse generating function is discontinuous at the CS domain boundary point (0, 0).
References	[EDM]

5.3.8.28 Lococentric Polar CS specification

Table 5.34 — Lococentric Polar CS

Element	Specification
Description	Localization of the Polar CS
CS label	LOCOCENTRIC_POLAR
CS code	27

Element	Specification
Function type	Generating function
CS descriptor	2D curvilinear
Properties	Orthogonal
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^2$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.
Coordinate-components	ρ: radius, and θ: angle in radians.
CS Domain	$\{(\rho,\theta) \text{ in } \mathbb{R}^2 \mid 0 < \rho \text{ and } 0 \leq \theta < 2\pi\} \cup \{(0,0)\}$
Generating function	$\boldsymbol{G}\big((\rho,\theta)\big) = \boldsymbol{L}_{\text{2D}} \circ \boldsymbol{G}_{\text{P}}\big((\rho,\theta)\big)$ $= [x,y]^{\text{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1$, $y = \boldsymbol{p} \bullet \boldsymbol{e}_2$, $\boldsymbol{p} = \boldsymbol{q} + \rho\cos(\theta)\,\boldsymbol{r} + \rho\sin(\theta)\,\boldsymbol{s}$, $\boldsymbol{L}_{\text{2D}} =$ the 3D localization operator, and $\boldsymbol{G}_{\text{P}} =$ the Cylindrical CS generating function
Domain of the inverse	$\mathbb{R}^2$
Inverse	$\boldsymbol{G}^{-1}([x,y]^{\text{T}}) = \boldsymbol{G}_{\text{P}}^{-1} \circ \boldsymbol{L}_{\text{2D}}^{-1}([x,y]^{\text{T}}) = (\rho,\theta)$, where: $\rho = \sqrt{u^2+v^2}$, $\theta = \begin{cases} \arctan2(v,u) & \text{if } v \geq 0 \\ 2\pi + \arctan2(v,u) & \text{if } v < 0, \end{cases}$ $u = \boldsymbol{p} \bullet \boldsymbol{r}$, $v = \boldsymbol{p} \bullet \boldsymbol{s}$, $\boldsymbol{p} = (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2) - \boldsymbol{q}$, $\boldsymbol{L}_{\text{2D}}^{-1} =$ the 2D localization inverse operator, and $\boldsymbol{G}_{\text{P}}^{-1} =$ the Polar CS inverse generating function
COM	n/a
Point distortion	n/a

Element	Specification
Figures	
Notes	1) The generating function is the composition of the generating function for the Polar CS (see Table 5.33) with the 2D localization operator (see 5.3.6.2). 2) An alternate CS parameter set is given by: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^2$, and $\boldsymbol{r}$: the primary axis direction unit vector in $\mathbb{R}^2$. $\boldsymbol{s}$ is then computed as $\boldsymbol{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{r}$.
References	[EDM]

5.3.8.29 Euclidean 1D CS specification

Table 5.35 — Euclidean 1D CS

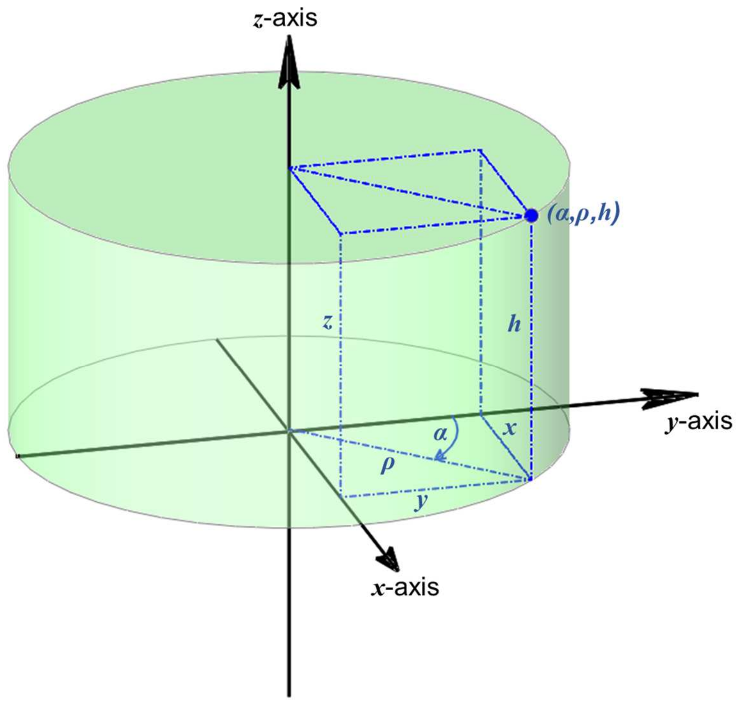
Element	Specification
Description	Euclidean 1D
CS label	EUCLIDEAN_1D
CS code	28
Function type	Generating function
CS descriptor	1D linear
Properties	none
CS parameters and constraints	none
Coordinate-components	u

Element	Specification
CS Domain	$\mathbb{R}^1$
Generating function	$\boldsymbol{G}_{\mathrm{E1D}}(u) = (x)$ where: $x = u$.
Domain of the inverse	$\mathbb{R}^1$
Inverse	$\boldsymbol{G}_{\mathrm{E1D}}^{-1}(x) = (u)$ where: $u = x$.
COM	n/a
Point distortion	n/a
Figures	none
Notes	1) Coordinate-space 1-tuples are identified with position-space 1-tuples. 2) This abstract coordinate system is also used to realize temporal coordinate systems (see 5.5).
References	[EDM]

5.3.8.30 Azimuthal Cylindrical CS specification

Table 5.36 — Azimuthal Cylindrical CS

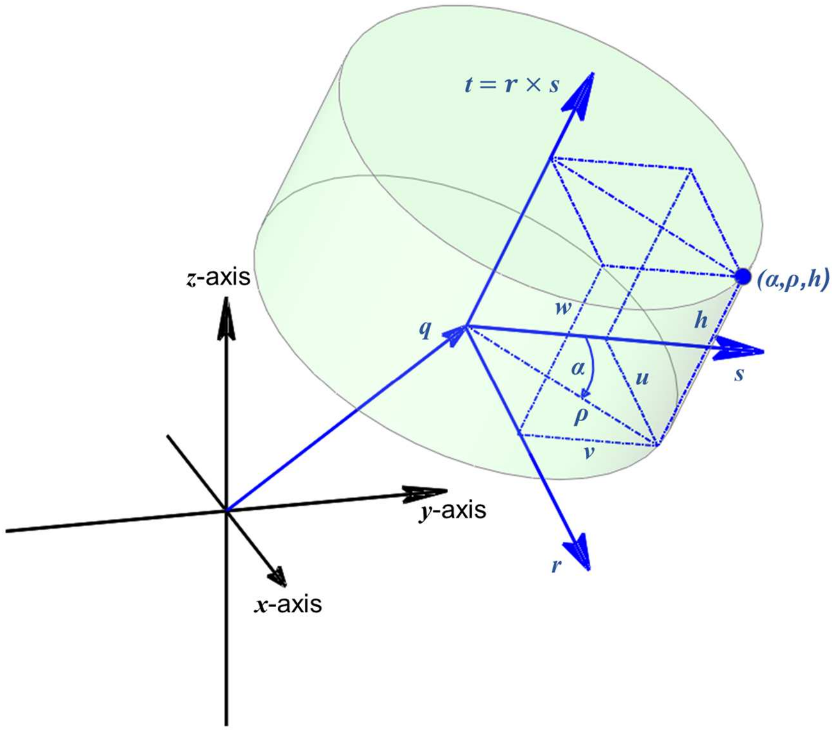
Element	Azimuthal Cylindrical
Description	Azimuthal Cylindrical
CS label	AZIMUTHAL_CYLINDRICAL
CS code	29
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	none
Coordinate-components	α: angle, in radians, ρ: radius, and h: height.
CS Domain	$\{(\alpha, \rho, h)$ in $\mathbb{R}^3$ $\vert 0 \leq \alpha < 2\pi$ and $\rho \geq 0\} \cup \{(0,0,h) \vert$ h in $\mathbb{R}\}$
Generating function	$\boldsymbol{G}(\alpha, \rho, h) = \begin{bmatrix} \rho \sin \alpha \\ \rho \cos \alpha \\ h \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}^{-1}\left([x, y, z]^{\mathrm{T}} \right) = (\alpha, \rho, h)$ where: $\alpha = \begin{cases} \mathrm{arctan2}(x, y), & \text{if } x \geq 0 \\ \mathrm{arctan2}(x, y) + 2\pi, & \text{if } x < 0 \end{cases}$ $\rho = \sqrt{x^2 + y^2}$ $h = z$

Element	Azimuthal Cylindrical
COM	n/a
Point distortion	n/a
Figures	
Notes	$\boldsymbol{F}^{-1}$ is discontinuous on the half-plane defined: all $[x, y, z]^{\mathrm{T}}$ in $\mathbb{R}^3$ that satisfy $x = 0$ and $y \geq 0$.
References	[EDM]

5.3.8.31 Lococentric Azimuthal Cylindrical CS specification

Table 5.37 — Lococentric Azimuthal Cylindrical CS

Element	Lococentric Azimuthal Cylindrical
Description	Localization of the Azimuthal Cylindrical CS
CS label	LOCOCENTRIC_AZIMUTHAL_CYLINDRICAL
CS code	30
Function type	Generating function
CS descriptor	3D curvilinear
Properties	Orthogonal
CS parameters and constraints	Localization parameters: $\boldsymbol{q}$: the lococentric origin in $\mathbb{R}^3$, and $\boldsymbol{r}$, $\boldsymbol{s}$: axis directions in $\mathbb{R}^3$. Constraints: $\boldsymbol{r}$ and $\boldsymbol{s}$ are orthonormal.

Element	Lococentric Azimuthal Cylindrical
Coordinate-components	α: angle, in radians, ρ: radius, and h: height.
CS Domain	$\{(\alpha,\rho,h)$ in $\mathbb{R}^3 \mid 0 \le \alpha < 2\pi$ and $\rho \ge 0\} \cup \{(0,0,h) \mid h$ in $\mathbb{R}\}$
Generating function	$\boldsymbol{G}(\alpha,\rho,h) = \boldsymbol{L}_{\mathrm{3D}} \circ \boldsymbol{G}_{\mathrm{AC}}\big((\alpha,\rho,h)\big) = [x,y,z]^{\mathrm{T}}$ where: $x = \boldsymbol{p} \bullet \boldsymbol{e}_1,\ \ y = \boldsymbol{p} \bullet \boldsymbol{e}_2,\ \ z = \boldsymbol{p} \bullet \boldsymbol{e}_3,$ $\boldsymbol{p} =\ \boldsymbol{q} + \rho\sin(\alpha)\,\boldsymbol{r} + \rho\cos(\alpha)\,\boldsymbol{s} + h\boldsymbol{r} \times \boldsymbol{s},$ $\boldsymbol{L}_{\mathrm{3D}} =$ the 3D localization operator, and $\boldsymbol{G}_{\mathrm{AC}} =$ the Azimuthal Cylindrical CS generating function.
Domain of the inverse	$\mathbb{R}^3$
Inverse	$\boldsymbol{G}^{-1}(\,[x,y,z]^{\mathrm{T}}\,) = \boldsymbol{G}_{\mathrm{AC}}^{-1} \circ \boldsymbol{L}_{\mathrm{3D}}^{-1}\,([x,y,z]^{\mathrm{T}}) = (\alpha,\rho,h)$ where: $\alpha = \begin{cases} \operatorname{arctan2}(u,v) & \text{if } u \ge 0 \\ 2\pi + \operatorname{arctan2}(u,v) & \text{if } u < 0 \end{cases}$ $\rho = \sqrt{u^2+v^2}$ $h = \boldsymbol{p} \bullet (\boldsymbol{r} \times \boldsymbol{s}),\ \ u = \boldsymbol{p} \bullet \boldsymbol{r},\ \ v = \boldsymbol{p} \bullet \boldsymbol{s},$ $u = \boldsymbol{p} \bullet \boldsymbol{r},\ \ v = \boldsymbol{p} \bullet \boldsymbol{s},$ $\boldsymbol{p} =\ (x\,\boldsymbol{e}_1 + y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3) - \boldsymbol{q},$ $\boldsymbol{L}_{\mathrm{3D}}^{-1} =$ the inverse 3D localization operator, and $\boldsymbol{G}_{\mathrm{AC}}^{-1} =$ the Azimuthal Cylindrical CS inverse generating function.
COM	n/a
Point distortion	n/a
Figures	
Notes	$\boldsymbol{G}^{-1}$ is discontinuous on the half-plane defined: all $[x,y,z]^{\mathrm{T}}$ in $\mathbb{R}^3$ that satisfy: $\{\,[x,y,z]^{\mathrm{T}} \mid ([x,y,z]^{\mathrm{T}} - \boldsymbol{q}) \bullet \boldsymbol{r} = 0,\ ([x,y,z]^{\mathrm{T}} - \boldsymbol{q}) \bullet \boldsymbol{s} \ge 0\}.$
References	[EDM]

5.4 Spatial coordinate systems

5.4.1 Introduction

A spatial coordinate system is an extension of an abstract coordinate system to object-space in that it assigns a unique coordinate n-tuple to each point in a region of an n-dimensional object-space. The generating function of an abstract coordinate system specifies vector values in terms of the position-space basis vectors. To extend the abstract coordinate system generating function to object-space, a specification of an orthonormal frame within the object-space is required. This frame can be specified with a normal embedding. A normal embedding is a length preserving isomorphism between the position-space and the object-space. This isomorphism provides the object-space an orthonormal frame termed the embedded frame (see 5.2.5). The spatial coordinate system assignment function is specified as the functional composition of the abstract coordinate system generating function followed by the normal embedding function. The abstract coordinate system generating function takes a coordinate to a position vector in position-space. The normal embedding then takes the position vector to its corresponding point in the embedded frame of the object-space, thus completing the coordinate to object-space point assignment. The advantage of this two-part approach is that an abstract coordinate system specification can be used with many different object-spaces of the same dimension, as well as with a single object-space with many different normal embeddings.

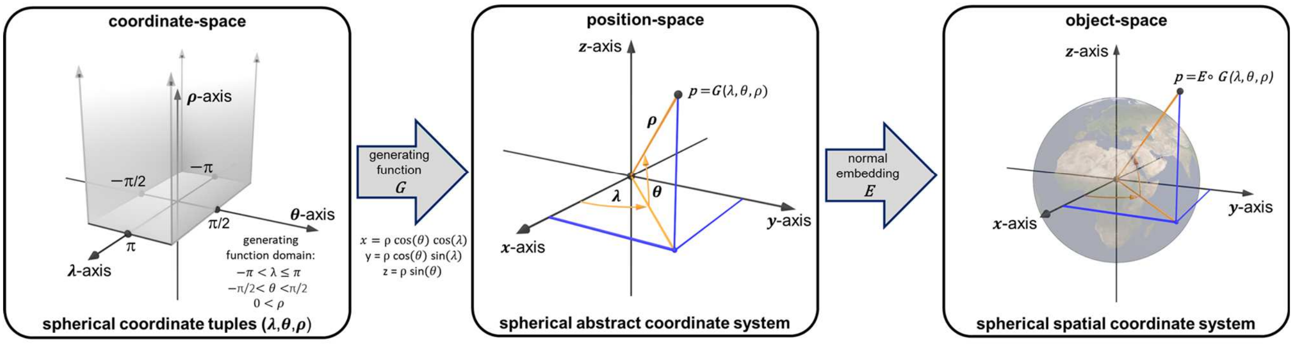


Figure 5.23 — Coordinate-space, position-space, and object-space relationships

Figure 5.23 illustrates the relationships between coordinate-space, position-space, object-space, the CS generating function, and a normal embedding for a spherical coordinate system.

5.4.2 Definition

A *spatial coordinate system* assigns a unique coordinate n-tuple to each point in a region of object-space. A spatial coordinate system has a domain in coordinate-space, a range in object-space, and a generating function that assigns a unique coordinate in the domain to each point in the object-space range.

This standard uses abstract CSs together with normal embeddings to create spatial CSs. Given an object-space and a normal embedding of position-space into that object-space, any abstract CS for a region of the position-space can be used to define a spatial CS with a generating function that takes the form:

$$\boldsymbol{p} = \boldsymbol{E} \circ \boldsymbol{G}(\boldsymbol{c})$$

where:
$\boldsymbol{c}$ is a coordinate in the CS domain,
$\boldsymbol{G}$ is the CS generating function,
$\boldsymbol{E}$ is the normal embedding function, and
$\boldsymbol{p}$ is the point in object-space associated with $\boldsymbol{c}$.

In the case where the abstract CS specification is parameterized (5.3.2), the CS parameter values shall be specified to complete the specification of $\boldsymbol{G}$. The composed function, $\boldsymbol{E} \circ \boldsymbol{G}$, is termed the spatial CS generating function.

Abstract CS types (5.3.3) and properties (5.3.5) are ascribed to a spatial CS as determined by the abstract CS. This method of creating spatial CSs has the advantage that an abstract coordinate system specification can be used with many different object-spaces of the same dimension, as well as with a single object-space with many different normal embeddings (see Example 1).

Localized frames and local tangent frames in object-space are defined for orthogonal spatial CSs as in 5.3.6.2 with the embedded frame replacing the role of position-space. Vector reference frames with respect to the embedded frame within object-space are similarly defined.

If an abstract CS with generating function $\boldsymbol{G}$ is localized with a localization operator $\boldsymbol{L}$ and localization parameters, $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, the resulting localized abstract coordinate system has generating function $\boldsymbol{L} \circ \boldsymbol{G}$ (see 5.3.6.2). When the normal embedding $\boldsymbol{E}$ is applied to such a localized abstract coordinate system, the resulting localized spatial coordinate system has spatial generating function $\boldsymbol{E} \circ \boldsymbol{L} \circ \boldsymbol{G}$. The localization parameters $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, are vectors in the canonical orthonormal frame of position-space. The embedding function $\boldsymbol{E}$ is an isomorphism between the position-space frame and the embedded frame in object-space, thus the object-space vectors, $\boldsymbol{E}(\boldsymbol{q}), \boldsymbol{E}(\boldsymbol{r}), \boldsymbol{E}(\boldsymbol{s})$, have the same coordinate-component values as the corresponding position-space vectors $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$.

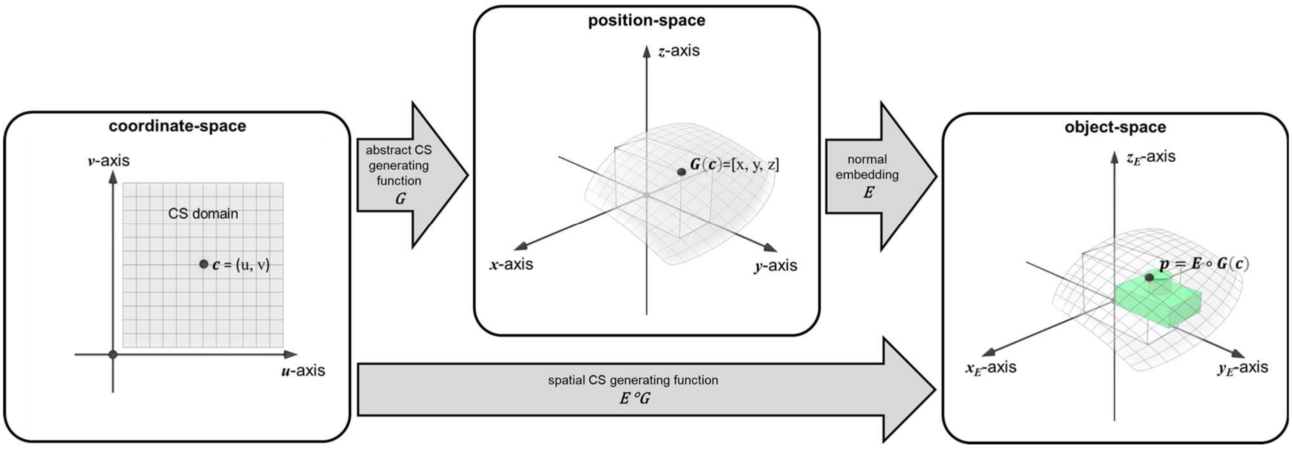


Figure 5.24 — A spatial CS of type surface

Figure 5.24 illustrates a spatial CS derived from a surface abstract CS and a 3D normal embedding. In this illustration, a surface coordinate (u, v) in coordinate-space is assigned to a position vector $[x, y, z]$ in position-space. Thatposition then identifies a location in the space of an object via the normal embedding of position-space. The normal embedding is determined by the selection of an origin and three orthogonal unit vectors in the object-space of the physical object. (The surface is not required to include any part of the physical object.)

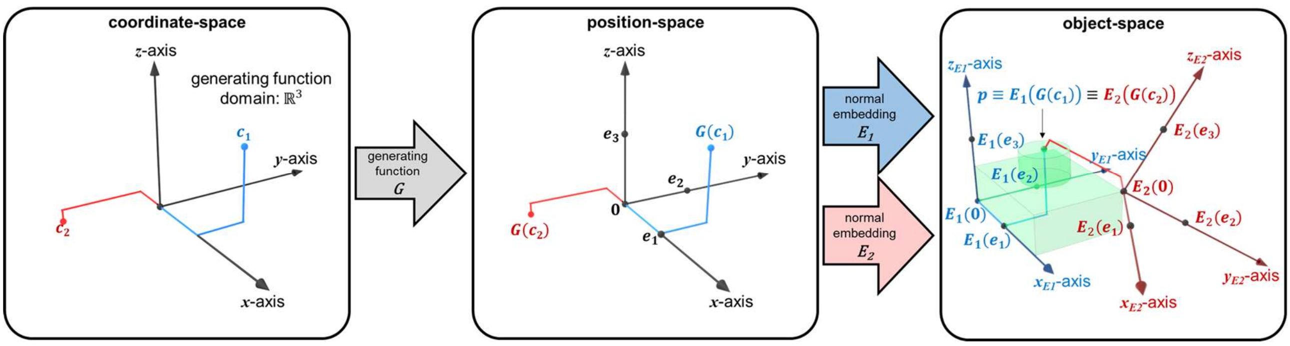


Figure 5.25 — Two spatial coordinate systems for the same object-space

EXAMPLE 1 Two coordinate tuples, c_1 and c_2, in the coordinate domain of an abstract Euclidean 3D CS are mapped by the generating function $\boldsymbol{G}$ into corresponding position vectors $\boldsymbol{G}(\boldsymbol{c}_1)$ and $\boldsymbol{G}(\boldsymbol{c}_2)$ in 3D position-space. $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are two distinct normal embeddings of 3D position-space into an object-space. One spatial Euclidean 3D CS is defined by the composition of the normal embedding $\boldsymbol{E}_1$ with the generating function $\boldsymbol{G}$. This composition $\boldsymbol{E}_1 \circ \boldsymbol{G}$ functionally maps the coordinate $\boldsymbol{c}_1$ to the point $\boldsymbol{p}$ in object-space. In this example, the second normal embedding $\boldsymbol{E}_2$ is also composed with generating function $\boldsymbol{G}$ to define a second distinct spatial Euclidean 3D CS. The composition $\boldsymbol{E}_2 \circ \boldsymbol{G}$ functionally maps $\boldsymbol{c}_2$ to the same point $\boldsymbol{p}$ in the object-space. Thus, the point $\boldsymbol{p}$ has Euclidean 3D coordinate $\boldsymbol{c}_1$ in the first spatial CS, and Euclidean 3D coordinate $\boldsymbol{c}_2$ in the second spatial CS. The relationship between the point $\boldsymbol{p}$ and the coordinates $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ is given by $\boldsymbol{p} \equiv \boldsymbol{E}_1(\boldsymbol{G}(\boldsymbol{c}_1)) \equiv \boldsymbol{E}_2(\boldsymbol{G}(\boldsymbol{c}_2))$. In Figure 5.25 the two embedded frames in object-space for the two normal embeddings are depicted in two different colours.

EXAMPLE 2 An engineering model is designed in abstract 3D Euclidean space using the Euclidean CS. In this simple case, coordinate-space, position-space, and object-space all have identical structure (the generating function and normal embedding are both the identity operator). However, each of the three spaces (coordinate-space, position-space, and object-space) serves a distinct role.

5.5 Temporal coordinate systems

5.5.1 Introduction

Time and position are often used together by an application to describe when a given condition exists or when an object was present at a given location. Furthermore, in dynamic physical systems, the normal embedding that maps position-space to an object-space may change over time. As a result, the relationship between coordinates and positions is time dependent. Thus, there is a requirement to identify time as well as position in environmental representation. In such systems, time, and time differences, must be taken into account in order to accurately determine positions and position differences.

This International Standard uses the concept of time in several ways. An object reference model (see 7.4) has either a static or dynamic binding to a spatial object. In the latter case, time is a parameter of the reference transformation (see 7.4.5) that specifies the binding (see 7.5). Spatial reference frames (see Clause 8) that are based on dynamic object reference models also depend on a time parameter. These dynamic cases reduce to the corresponding static cases by fixing a value for the time parameter.

Time also plays a role in static object reference model bindings that are based on physical measurements of objects or systems that change with time. A time stamp is used to identify the epoch for which these measurements are applicable.

A *temporal coordinate system* is a realization of an abstract Euclidean 1D coordinate system (see Table 5.35) that assigns a one-to-one monotonically increasing relationship between temporal coordinate values and instants in time, such that larger coordinate values are assigned to later instants in time. A temporal coordinate system is used to associate a unique time with an event or reference.

An *integrated temporal coordinate system* is a type of temporal coordinate system that fixes an origin (termed the *epoch*) and then continuously integrates duration units as they occur. An integrated temporal coordinate system is based on a duration unit derived from a periodic physical phenomenon, such as an atomic resonance frequency.

A *dynamic temporal coordinate system* is a type of temporal coordinate system that is based on a mathematical model of a dynamic physical system, typically planetary motion, where time is one of the parameters that unambiguously identifies the configuration of the system. The initial conditions of the physical system specify the origin epoch. Observed configurations of the physical system are associated with the time parameter of the mathematical model to specify the dynamic temporal coordinate system.

An integrated temporal coordinate system differs from a dynamic temporal coordinate system in that the former accumulates the duration of a periodic phenomenon while the latter ties a mathematical model parameter to the state of a physical system.

5.5.2 Universal time

Universal time (UT) is a general designation of a set of dynamic temporal coordinate systems based on the rotation of the Earth. There are different forms of UT whose values may differ by a few hundredths of a second:

a) *Universal Time observed* (UT0) is the mean solar time at the prime meridian obtained from direct astronomical observation.

b) *Universal Time polar motion corrected* (UT1) is UT0 corrected for the effects of small movements of the Earth relative to the axis of rotation (polar variation).

c) *Universal Time Earth rotation corrected* (UT2) is UT1 corrected for the effects of a small seasonal fluctuation in the rate of rotation of the Earth.

Complete definitions of UT0, UT1, UT2, and the concepts involved in their definitions may be found in the publications of the International Earth Rotation and Reference Systems Service [IERS] that maintains these three temporal coordinate systems.

5.5.3 International atomic time

International atomic time (TAI) is the integrated temporal coordinate system with the SI second as unit of duration and origin epoch defined so that the difference between UT1 and TAI is zero on 1 January 1958. TAI is maintained by the Bureau International des Poids et Mesures (International Bureau of Weights and Measures) (BIPM) and is generated by collecting and combining the data from a worldwide ensemble of atomic clocks.

5.5.4 Coordinated universal time

Coordinated universal time (UTC) is a temporal coordinate system that is based on both TAI and UT1. UTC is specified by the Radio Communication Bureau of the International Telecommunication Union (ITU-R) in publication TF.460-6:2002 [I460]. It is a compromise between highly stable TAI and irregular UT. UTC is maintained by the BIPM with assistance from the IERS. As currently defined, UTC runs at the same rate as TAI but is adjusted by the insertion or deletion of seconds (termed positive or negative leap-seconds) to ensure approximate agreement with UT1 to within 0,9 seconds, which is sufficient for purposes of astronomical navigation. As a consequence, UTC and TAI differ by an integer number of seconds. On 01 January 2022, TAI was 37 seconds ahead of UTC.

5.5.5 Specified temporal coordinate systems

This International Standard provides standard codes and labels to identify temporal coordinate systems. The codes and labels are defined by specifying all elements presented in Table 5.38. The standard codes and labels are specified in Table 5.39. Additional temporal coordinate systems may be specified by registration in accordance with Clause 13.

Table 5.38 — Temporal coordinate system specification elements

Element	Definition
Temporal CS label	The label (see 13.2.2) for the temporal coordinate system.
Temporal CS code	The code (see 13.2.3) for the temporal coordinate system.
Description	A description of the temporal coordinate system, including any common name.
Reference(s)	The reference(s) (see 13.2.5) for the definitions of the temporal coordinate system.

Table 5.39 — Temporal coordinate system specifications

Temporal CS label	Temporal CS code	Description	Reference(s)
TAI	1	International atomic time (TAI)	[I460]
UTC	2	Coordinated universal time (UTC)	[I460]

In this International Standard, times and dates refer to UTC unless explicitly indicated otherwise. Whenever a temporal coordinate system other than UTC is required, that temporal coordinate system shall be explicitly specified. Local realizations of UTC may differ by tens of nanoseconds so that real-time applications that require nanosecond accuracy shall reference the appropriate realization of UTC.