Annex A
(normative)

Mathematical foundations

A.1 Introduction

This annex identifies the concepts from mathematics used in this International Standard and specifies the notation used for those concepts. A reader of this International Standard is assumed to be familiar with mathematics including set theory, linear algebra, and the calculus of several real variables as presented in reference works such as the *Encyclopedic Dictionary of Mathematics* [EDM].

A.2 $\mathbb{R}^n$ as a real vector space

An ordered set of n real numbers $\boldsymbol{a}$ where n is a natural number is termed an *n-tuple of real numbers* and shall be denoted by $\boldsymbol{a} = [a_1, a_2, a_3, \cdots, a_n]$ The set of all n-tuples of real numbers is denoted by $\mathbb{R}^n$. $\mathbb{R}^n$ is an n-dimensional vector space.

The *canonical basis* for $\mathbb{R}^n$ is defined as:

$$\boldsymbol{e}_1 = [1,0,\cdots,0],\ \boldsymbol{e}_2 = [0,1,\cdots,0],\cdots,\ \boldsymbol{e}_n = [0,0,\cdots,1].$$

The elements of $\mathbb{R}^n$may be termed *points* or *vectors*. The latter term is used in the context of directions or vector space operations.

The zero vector $[0,0,\cdots,0]$ is denoted by **0**.

Definitions A.2(a) through A.2(j) apply to any vectors $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$ and $\boldsymbol{y} = [y_1, y_2, \cdots, y_n]$ in $\mathbb{R}^n$:

a) The *inner product* or *dot-product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as:

$$\boldsymbol{x} \bullet \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

b) Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are termed *orthogonal* if $\boldsymbol{x} \bullet \boldsymbol{y} = 0$.

c) If $n \geq 2$, two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are termed *perpendicular* if and only if they are orthogonal.

NOTE 1 If $n \geq 2$, $\boldsymbol{x} \bullet \boldsymbol{y} = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos(\alpha)$ where α is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

d) $\boldsymbol{x}$ is termed *orthogonal to a set* of vectors if $\boldsymbol{x}$ is orthogonal to each vector that is a member of the set.

e) The *norm* of $\boldsymbol{x}$ is defined as

$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \bullet \boldsymbol{x}}.$$

NOTE 2 The norm of $\boldsymbol{x}$ represents the length of the vector $\boldsymbol{x}$. Only the zero vector **0** has norm zero.

f) $\boldsymbol{x}$ is termed a *unit vector* if $\|\boldsymbol{x}\| = 1$.

g) A set of two or more orthogonal unit vectors is termed an *orthonormal set of vectors*.

EXAMPLE The canonical basis is an example of an orthonormal set of vectors.

h) The *Euclidean metric* d is defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

i) The value of $d(\boldsymbol{x}, \boldsymbol{y})$ is termed the *Euclidean distance* between $\boldsymbol{x}$ and $\boldsymbol{y}$.

j) The *cross product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^3$ is defined as the vector:

$$\boldsymbol{x} \times \boldsymbol{y} = [x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1].$$

NOTE 3 The vector $\boldsymbol{x} \times \boldsymbol{y}$ is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$, and

$$\|\boldsymbol{x} \times \mathbf{y}\| = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \sin(\alpha),$$

where α is the angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

A.3 The point set topology of $\mathbb{R}^n$

Given a point $\boldsymbol{p}$ in $\mathbb{R}^n$ and a real value $\varepsilon > 0$, the set $\{\boldsymbol{q}\ in\ \mathbb{R}^n | d(\boldsymbol{p}, \boldsymbol{q}) < \varepsilon\}$ is termed the *ε-neighbourhood* of $\boldsymbol{p}$.

Given a set $D \subset \mathbb{R}^n$ and a point $\boldsymbol{p}$, the following terms are defined:

a) $\boldsymbol{p}$ is an *interior point of* D if at least one ε-neighbourhood of $\boldsymbol{p}$ is a subset of D.

b) The *interior of a set* D is the set of all points that are interior points of D.

NOTE 1 The interior of a set may be empty.

c) D is *open* if each point of D is an interior point of D. Consequently, D is open if it is equal to its interior.

d) $\boldsymbol{p}$ is a *closure point of* D if every ε-neighbourhood of $\boldsymbol{p}$ has a non-empty intersection with D.

NOTE 2 Every member of D is a closure point of D.

e) The *closure of a set* D is the set of all points that are closure points of D.

f) D is a *closed set* if it is equal to the closure set of D.

g) A set D is *replete* if all points in D belong to the closure of the interior of D.

NOTE 3 Every open set is replete. The union of an open set with any or all of its closure points forms a replete set. In particular, the closure of an open set is replete.

EXAMPLE 1 In $\mathbb{R}^2$ $\{(\boldsymbol{x}, \boldsymbol{y}) | -\pi < \boldsymbol{x} < \pi, -\pi/2 < \boldsymbol{y} < \pi/2\}$ is open and therefore replete.

EXAMPLE 2 $\{(\boldsymbol{x}, \boldsymbol{y}) | -\pi < \boldsymbol{x} \leq \pi, -\pi/2 < \boldsymbol{y} < \pi/2\}$ is replete.

EXAMPLE 3 $\{(\boldsymbol{x}, \boldsymbol{y}) | -\pi < \boldsymbol{x} \leq \pi, -\pi/2 \leq \boldsymbol{y} \leq \pi/2\}$ is closed and replete.

A.4 Smooth functions on $\mathbb{R}^n$

A real-valued function f defined on a replete domain in $\mathbb{R}^n$ is termed *smooth* if it is continuous and its first derivative exists and is continuous at each point in the interior of its domain.

The *gradient* of f is the vector of first order partial derivatives

$$\mathbf{grad}(f) = \left[\frac{\partial f}{\partial v_1}, \frac{\partial f}{\partial v_2}, \cdots, \frac{\partial f}{\partial v_n}\right].$$

Definitions A.4(a) through A.4(g) apply to any vector-valued function $\boldsymbol{F}$ defined on a replete domain D in $\mathbb{R}^n$ with range in $\mathbb{R}^m$.

a) The j^{th}-*component function* of a vector-valued function $\boldsymbol{F}$ is the real-valued function f_j defined by $f_j = \boldsymbol{e}_j \bullet \boldsymbol{F}$ where $\boldsymbol{e}_j$ is the j^{th} canonical basis vector, $j = 1, 2, \cdots, m$.

In this case:

$$\boldsymbol{F}(\boldsymbol{v}) = [f_1(\boldsymbol{v}), f_2(\boldsymbol{v}), f_3(\boldsymbol{v}), \ldots, f_m(\boldsymbol{v})] \text{ for } \boldsymbol{v} = [v_1, v_2, v_3, \ldots, v_n] \text{ in } D.$$

b) $\boldsymbol{F}$ is termed *smooth* if each component function f_j is smooth.

c) The *first derivative* of a smooth vector-valued function $\boldsymbol{F}$, denoted $\partial\boldsymbol{F}$, evaluated at a point in the domain is the $n \times m$ matrix of partial derivatives evaluated at the point:

$$\left[\frac{\partial f_j}{\partial v_i}\right] \quad i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m.$$

d) The *Jacobian matrix* of $\boldsymbol{F}$ at the point $\boldsymbol{v}$ is the matrix of the first derivative of $\boldsymbol{F}$.

NOTE 1 The rows of the Jacobian matrix are the gradients of the component functions of $\boldsymbol{F}$.

e) In the case $m = n$, the Jacobian matrix is square and its determinant is termed the *Jacobian determinant.*

f) In the case $m = n$, $\boldsymbol{F}$ is termed *orientation preserving* if its Jacobian determinant is strictly positive for all points in D.

g) A vector-valued function $\boldsymbol{F}$ defined on $\mathbb{R}^n$ is *linear* if:

$$\boldsymbol{F}(a\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{y}) \text{ for all real scalars } a \text{ and vectors } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ in } \mathbb{R}^n$$

NOTE 2 All linear functions are smooth.

A vector-valued function $\boldsymbol{E}$ defined on $\mathbb{R}^n$ is *affine* if $\boldsymbol{F}$, defined by $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}(\boldsymbol{0})$, is a linear function. All affine functions on $\mathbb{R}^n$ are smooth.

A function may be alternatively termed an *operator* especially when attention is focused on how the function maps a set of points in its domain onto a corresponding set of points in its range.

EXAMPLE The localization operators (see 5.3.6.2).

A.5 Functional composition

If $\boldsymbol{F}$ and $\boldsymbol{G}$ are two vector valued functions and the range of $\boldsymbol{G}$ is contained in the domain of $\boldsymbol{F}$, then $\boldsymbol{F} \circ \boldsymbol{G}$, the *composition* of $\boldsymbol{F}$ with $\boldsymbol{G}$, is the function defined by $\boldsymbol{F} \circ \boldsymbol{G}(\boldsymbol{x}) \equiv \boldsymbol{F}(\boldsymbol{G}(\boldsymbol{x}))$. $\boldsymbol{F} \circ \boldsymbol{G}$ has the same domain as $\boldsymbol{G}$, and the range of $\boldsymbol{F} \circ \boldsymbol{G}$ is contained in the range of $\boldsymbol{F}$.

Functional composition also applies to scalar-valued functions f and g, If the range of g is contained in the domain of f, then $f \circ g(x)$, the composition of f with g, is the function defined by $f \circ g(x) \equiv f(g(x))$.

A.6 Smooth surfaces in $\mathbb{R}^3$

A.6.1 Implicit definition

A *smooth surface* in $\mathbb{R}^3$ is *implicitly* specified by a real-valued smooth function f defined on $\mathbb{R}^3$ as the set S of all points (x, y, z) in $\mathbb{R}^3$ satisfying:

a) $f(x, y, z) = 0$ and

b) $\mathbf{grad}(f)(x, y, z) \neq \mathbf{0}$.

In this case, f is termed a *surface generating function* for the surface S.

EXAMPLE 1 If $\boldsymbol{n} \neq \mathbf{0}$ and $\boldsymbol{p}$ are vectors in $\mathbb{R}^3$ and $f(v) = \boldsymbol{n} \bullet (\boldsymbol{v} - \boldsymbol{p})$, then f is smooth and $\mathbf{grad}(f) = \boldsymbol{n} \neq 0$. The plane which is perpendicular to $\boldsymbol{n}$ and contains $\boldsymbol{p}$ is the smooth surface implicitly defined by the surface generating function f.

Special cases:

When $\boldsymbol{n} = (1, 0, 0)$ and $\boldsymbol{p} = \mathbf{0}$, the yz-plane is implicitly defined.

When $\boldsymbol{n} = (0, 1, 0)$ and $\boldsymbol{p} = \mathbf{0}$, the xz-plane is implicitly defined.

When $\boldsymbol{n} = (0, 0, 1)$ and $\boldsymbol{p} = \mathbf{0}$, the xy-plane is implicitly defined.

The *surface normal* $\boldsymbol{n}$ at a point $\boldsymbol{p} = (x, y, z)$ on the surface implicitly specified by a surface generating function f is defined as:

$$\boldsymbol{n} \equiv \frac{1}{\|\mathbf{grad}(f)(\boldsymbol{p})\|}\mathbf{grad}(f)(\boldsymbol{p}).$$

NOTE $-\boldsymbol{n}$ is also a surface normal to S at $\boldsymbol{p}$. The surface generating function f determines the surface normal direction: $\boldsymbol{n}$ or $-\boldsymbol{n}$.

The *tangent plane* to a surface at a point $\boldsymbol{p} = (x, y, z)$ on the surface S implicitly defined by a surface generating function f is the plane which is the smooth surface implicitly defined by $h(v) = \boldsymbol{n} \bullet (\boldsymbol{v} - \boldsymbol{p})$ where $\boldsymbol{n}$ is the surface normal to S at $\boldsymbol{p}$.

EXAMPLE 2 If a and b are positive non-zero scalars, define

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1.$$

Then f is smooth and

$$\mathbf{grad}(f)(x, y, z) = \left(\frac{2x}{a^2}, \frac{2y}{a^2}, \frac{2z}{b^2}\right)$$

is never (0, 0, 0) on the surface implicitly specified by the set satisfying $f = 0$.

A.6.2 Ellipsoid surfaces

If a and b are positive non-zero scalars, the smooth function:

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1$$

is a surface generating function for an *ellipsoid of revolution* smooth surface S.

When $b \leq a$, the surface is termed an *oblate ellipsoid*. In this case a is termed the *major semi-axis*[31] of the oblate ellipsoid and b is termed the *minor semi-axis* of the oblate ellipsoid.

The *flattening* of an oblate ellipsoid is defined as $f = \frac{a-b}{a}$.

The *eccentricity* of an oblate ellipsoid is defined as $\varepsilon = \sqrt{1 - (b/a)^2}$.

The *second eccentricity* of an oblate ellipsoid is defined as $\varepsilon' = \sqrt{(a/b)^2 - 1}$.

When $b = a$, the oblate ellipsoid may be termed a *sphere* of radius $r = b = a$.

When $a < b$, the surface is termed a *prolate ellipsoid*. In this case, a is termed the *minor semi-axis* of the prolate ellipsoid and b is termed the *major semi-axis* of the prolate ellipsoid.

NOTE 1 A sphere of radius r is also implicitly defined by the surface generating function $f(x, y, z) = x^2 + y^2 + z^2 - r^2$.

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[31] a is half the length of the major axis. ISO 19111 labels the symbol a as the semi-major axis.

NOTE 2 The term spheroid is often used to denote an oblate ellipsoid with an eccentricity close to zero ("almost spherical").

A.7 Smooth curves in $\mathbb{R}^n$

A.7.1 Parametric definition

A.7.1.1 Smooth curve

A *smooth curve* in $\mathbb{R}^n$ is *parametrically* specified by a smooth one-to-one $\mathbb{R}^n$ valued function $\boldsymbol{F}(t)$ defined on a replete interval I in $\mathbb{R}$ such that $\|\partial \boldsymbol{F}(t)\| \neq 0$, for any t in I.

EXAMPLE 1 If $\boldsymbol{p}$ and $\boldsymbol{n}$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{n} \neq \boldsymbol{0}$ and $\boldsymbol{L}(t) = \boldsymbol{p} + t\,\boldsymbol{n}, -\infty < \boldsymbol{t} < +\infty$, then $\boldsymbol{L}$ is smooth and $\|\partial \boldsymbol{L}(t)\| = \|\boldsymbol{n}\| > 0$. The line which is parallel to $\boldsymbol{n}$ and which contains $\boldsymbol{p}$ is a smooth curve parametrically specified by $\boldsymbol{L}$.

EXAMPLE 2 If a and b are positive non-zero scalars and $b \leq a$, define

$\boldsymbol{F}(t) = (a\cos(t)\,, b\sin(t))$ for all t in the interval $-\pi < t \leq \pi$.

Then $\boldsymbol{F}$ is smooth and $\|\partial \boldsymbol{F}(t)\| \geq b > 0$ for all t in the interval and therefore parametrically specifies a smooth curve in $\mathbb{R}^2$.

An *ellipse* in $\mathbb{R}^2$ with major semi-axis a and minor semi-axis b, $0 < b \leq a$, is parametrically specified by:

$\boldsymbol{F}(t) = (a\cos(t)\,, b\sin(t))$ for all t in the interval $-\pi < t \leq \pi$.

A.7.1.2 Tangent to a smooth curve

If $\boldsymbol{C}(t)$ parametrically specifies a smooth curve C passing through a point $\boldsymbol{p} = \boldsymbol{C}(t_p)$, the *tangent vector* to C at $\boldsymbol{p}$ shall be defined as:

$$\boldsymbol{t} = \frac{1}{\|\mathbf{d}\boldsymbol{C}(t_p)\|}\mathbf{d}\boldsymbol{C}(t_p)$$

where $\mathbf{d}\boldsymbol{C}(t_p) = [\mathrm{d}C_1/\mathrm{d}t,\ \mathrm{d}C_2/\mathrm{d}t, \ldots, \mathrm{d}C_n/\mathrm{d}t\ ]$ is the first derivative of $\boldsymbol{C}$ evaluated at t_p.

NOTE $-\boldsymbol{t}$ is also a tangent vector to C at $\boldsymbol{p}$. The parameterization function $\boldsymbol{C}(t)$ determines the tangent vector direction: t or $-\boldsymbol{t}$.

A locus of points is a *directed curve* if it is the range of a smooth curve.

The *tangent line* to the curve C at $\boldsymbol{p}$ is a smooth curve parametrically specified by $\boldsymbol{T}(s) = \boldsymbol{p} + s\,\boldsymbol{t}, -\infty < s < +\infty$, where $\boldsymbol{t}$ is a tangent vector to C at $\boldsymbol{p}$. See Figure A.1.

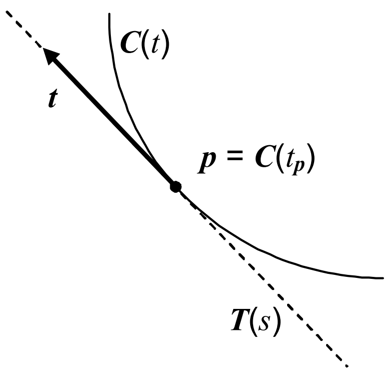

Figure A.1 — Tangent to a curve

A.7.1.3 Angle between curves

If two parametrically specified smooth curves C_1 and C_2 intersect at a point $\boldsymbol{p}$ then the *angle at* $\boldsymbol{p}$ *from* C_1 *to* C_2 is defined as the angle from the tangent vector $\boldsymbol{t}_1$ to the tangent vector $\boldsymbol{t}_2$ of the two curves, respectively, at $\boldsymbol{p}$. This is illustrated in Figure A.2.

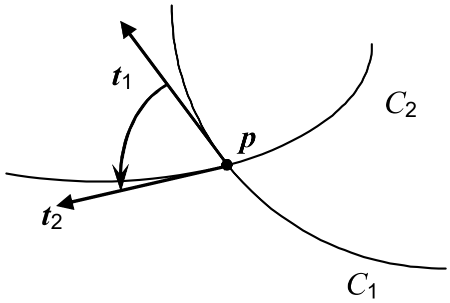

Figure A.2 — Angle between two curves

If a smooth curve C passes through a non-polar point $\boldsymbol{p}$ on an ellipsoid and the meridian at $\boldsymbol{p}$ is parameterized to start at the south pole and end at the north pole, then the *azimuth of* C *at* $\boldsymbol{p}$ is the clockwise angle at $\boldsymbol{p}$ from the meridian to C.

A.7.1.4 Closed curve

If a smooth function $\boldsymbol{F}$ is defined on a closed and bounded interval I with interval end points t_0 and t_1 and if $\boldsymbol{F}$ parametrically specifies a smooth curve on the interior of I and $\boldsymbol{p} = \boldsymbol{F}(t_0) = \boldsymbol{F}(t_1)$, then $\boldsymbol{F}$ generates a *closed curve* through $\boldsymbol{p}$.

EXAMPLE $\boldsymbol{F}(t) = (a\cos(t), b\sin(t))$, for all t in the interval $-\pi + \theta \le t \le \pi + \theta$.

If a and b are positive non-zero scalars and θ is given, $\boldsymbol{F}$ generates a closed curve though $\boldsymbol{p} = (a\cos(\pi + \theta), b\sin(\pi + \theta))$.

A.7.1.5 Surface curves, connected and orientable surfaces

If C is a smooth curve in $\mathbb{R}^3$ parametrically specified by $\boldsymbol{F}$ on the interval I and if S is a smooth surface generated by a surface generating function g, then C is a *surface curve in* S if $g \circ \boldsymbol{F}(t) = 0$ for all t in I. In this case C shall be said to lie in S.

EXAMPLE 1 If S is a smooth surface with generating function g and if $\boldsymbol{C}(s)$ defines a surface curve in S which passes through $\boldsymbol{p} = \boldsymbol{C}(s_p)$, then the tangent line to the curve at $\boldsymbol{p}$, $\boldsymbol{T}(s) = \boldsymbol{p} + s\,\mathbf{d}\boldsymbol{C}(t_p)$, lies[32] in the tangent plane to the surface S at $\boldsymbol{p}$. This is illustrated in Figure A.3.

[32] Since $g \circ \boldsymbol{C}(t) = 0$, the chain rule implies that $\mathbf{grad}(g) \bullet d\boldsymbol{C} = \mathrm{d}(g \circ \boldsymbol{C}(t))/\mathrm{d}t = 0$, so that $\boldsymbol{n} \bullet \mathbf{d}\boldsymbol{C} = 0$, where $\boldsymbol{n}$ is the surface normal at $\boldsymbol{p}$. $h(\boldsymbol{v}) = \boldsymbol{n} \bullet (\boldsymbol{v} - \boldsymbol{p})$ defines the tangent plane to the surface S at $\boldsymbol{p}$. $h(\boldsymbol{T}(s)) = h\left(\boldsymbol{p} + sd\boldsymbol{C}(t_{\boldsymbol{p}})\right) = \boldsymbol{n} \bullet (\boldsymbol{p} + sd\boldsymbol{C}(t_{\boldsymbol{p}}) - \boldsymbol{p}) = s(\boldsymbol{n} \bullet d\boldsymbol{C}) = 0$ so the tangent line lies in the tangent plane.

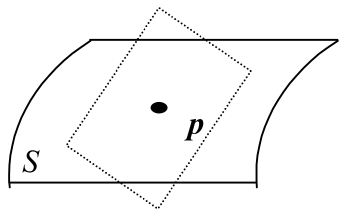

Figure A.3 — Tangent plane to a surface

A smooth surface S is *connected* if for any two distinct points in S, there exists a smooth surface curve parametrically specified by a smooth function defined on a bounded interval that lies in S and that contains the two points on the curve.

A connected surface S is termed an *orientable surface* if the normal vector at an arbitrary point $\boldsymbol{p}$ on S can be continued in a unique and continuous manner to the entire surface. A normal vector at a fixed point $\boldsymbol{p}_0$ may be *continued* if there does not exist a closed curve C in S through $\boldsymbol{p}_0$ such that the normal vector direction reverses when it is displaced continuously from $\boldsymbol{p}_0$ along C and back to $\boldsymbol{p}_0$.

An *oriented surface* is an orientable surface in which one side has been designated as positive.

EXAMPLE 2 If S is implicitly defined by $f = 0$, the side bounding the set satisfying $f > 0$ is designated as the positive side.

EXAMPLE 3 A Möbius strip is an example of a non-orientable surface.

NOTE If S is implicitly specified, it is an orientable surface[33].

A.7.2 Implicit definition

A *smooth curve* in $\mathbb{R}^2$ may be *implicitly* specified by a real-valued smooth function f on $\mathbb{R}^2$ as the set S of all points (x, y) in $\mathbb{R}^2$ satisfying:

a) $f(x,y) = 0$ and

b) $\mathbf{grad}(f)(x,y) \neq [0,0]$.

In this case, f is termed a *curve generating function* for the curve C.

EXAMPLE If a and b are positive non-zero scalars, define

$$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

.

Then f is smooth and

$$\mathbf{grad}(f)(x,y) = \left[\frac{2x}{a^2}, \frac{2y}{b^2}\right]$$

is never (0, 0) on the curve $f = 0$.

If $0 < b \leq a$, an *ellipse* in $\mathbb{R}^2$ with major semi-axis a and minor semi-axis b, is *implicitly* specified by the curve generating function defined by:

$$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

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33 Since a surface generating function for S is smooth, its gradient is continuous. Therefore the surface normal will be a continuous function of points on a curve that lies in S.

A.7.3 Arc length

If $\boldsymbol{p} = \boldsymbol{C}(t_p)$ and $\boldsymbol{q} = \boldsymbol{C}(t_q)$ are two points on a smooth surface curve defined by $\boldsymbol{C}$ and $t_p < t_q$, the *arc length* of the curve segment with endpoints $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined by the quantity:

$$\int_{t_p}^{t_q} \|\mathbf{d}\boldsymbol{C}(t)\| \ \mathrm{d}t.$$

A.7.4 Geodesics on an ellipsoid

There are several equivalent ways to define geodesics. This definition is specific to ellipsoids. Using the surface geodetic coordinate system on an oblate ellipsoid, a smooth surface curve $\big(\lambda(s), \varphi(s)\big)$ parameterized by arc length s, is a *geodesic* if and only if it satisfies the following three differential equations:

$$\frac{d\varphi}{ds} = \frac{\cos\alpha}{R_M(\varphi)},$$
$$\frac{d\lambda}{ds} = \frac{\sin\alpha}{R_N(\varphi)\cos\varphi}, \text{ and}$$
$$\frac{d\alpha}{ds} = \sin\varphi \frac{d\lambda}{ds},$$

where α is the azimuth of the curve at the point $[\lambda(s), \varphi(s)]$, $\mathcal{M}$ is the radius of curvature in the meridian, and $\mathcal{N}$ is the radius of curvature in the prime vertical (functions $\mathcal{M}$ and $\mathcal{N}$ are defined in Table 5.6.).

Every smooth surface curve in an oblate ellipsoid surface satisfies the first two equations. The third equation, known in geodesy as Bessel's equation, is a necessary and sufficient condition for a smooth surface curve to be a geodesic (see [RAPP1]).

A.8 Special functions

A.8.1 Double argument arctangent function

The two-argument form of inverse tangent, $\arctan2(y, x)$, returns a value adjusted by the quadrant of the point (x, y). Given real numbers x and y,

$\arctan2(y, x) = \theta$

where θ is the unique value satisfying $-\pi < \theta \leq \pi$ and

if $r = 0$,

$\theta = 0$, else

if $r > 0$,

$x = r \cos \theta$, and

$y = r \sin \theta$

where:

$r = \sqrt{x^2 + y^2}$.

NOTE If $x > 0$, then $\arctan2(y, x) = \arctan(y)/x$ *"principal value."* Some software implementation libraries reverse the roles of x and y.

A.8.2 Jacobian elliptic functions

Jacobian elliptic functions are defined in terms of certain elliptic integrals. There are many equivalent definitions, each involving special notation (see [ABST]). The notation used in this International Standard is given here.

The elliptic integral of the first kind is defined by:

$$f(\varphi|\varepsilon^2) = \int_0^{\varphi} \frac{d\xi}{\sqrt{1 - \varepsilon^2 \, sin^2(\xi)}},$$

and $f^{-1}(u|\varepsilon\text{^}2)$ is its inverse.

the *Jacobian elliptic functions* used in this International Standard are defined by,

If $u = f(\varphi|\varepsilon^2)$,

$\mathrm{sn}(u|\varepsilon^2) = \sin(\varphi)$,

$\mathrm{cn}(u|\varepsilon^2) = \cos(\varphi)$, and

$\mathrm{dn}(u|\varepsilon^2) = \sqrt{1 - \varepsilon^2 \sin^2(\varphi)}$

where: $\varphi = f^{-1}(u|\varepsilon^2)$.

Series expansions for these Jacobian elliptic functions are given in [ABST].

NOTE The complex functions $"sn"(u|\varepsilon^2), "cn"(u|\varepsilon^2),$ and $“dn”(u|\varepsilon^2)$ are termed Jacobian elliptic functions in [ABST] and [DOZI] and are termed Jacobi functions in [LLEE].

A.9 Projection function

A.9.1 Geometric projection functions into a developable surface

A *projection function* in $\mathbb{R}^3$ is a smooth function defined on a connected replete domain in $\mathbb{R}^3$onto a surface in the domain whose points are all fixed points of the function. Projection functions defined below project their domain onto such a plane, cone, or cylinder surface and are classified as planar, conic, or cylindrical projection functions according to the class of the fixed-point surface.

NOTE Some map projections CSs (see 5.3.7) are unrelated to any geometric projection.

A.9.2 Planar projection functions

A.9.2.1 Orthographic projection function

Given a plane in $\mathbb{R}^3$, the domain of the *orthographic projection function* is either all $\mathbb{R}^3$or the half space on one side of (and including) the plane. Given a point $\boldsymbol{x}$ in the domain, if $\boldsymbol{x}$ is not in the plane, there is one line that both passes through $\boldsymbol{x}$ and is perpendicular to the plane. If $\boldsymbol{p}$ is the point at the intersection of that line with the plane, the projection $\boldsymbol{F}$ assigns the value $\boldsymbol{p}$ to $\boldsymbol{x}$. That is $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{p}$. If the point $\boldsymbol{x}$ lies in the plane, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$ so that points in the plane are fixed points of the projection. In the case that the plane is the xy-plane, $\boldsymbol{F}(x, y, z) = (x, y, 0)$. See Figure A.4.

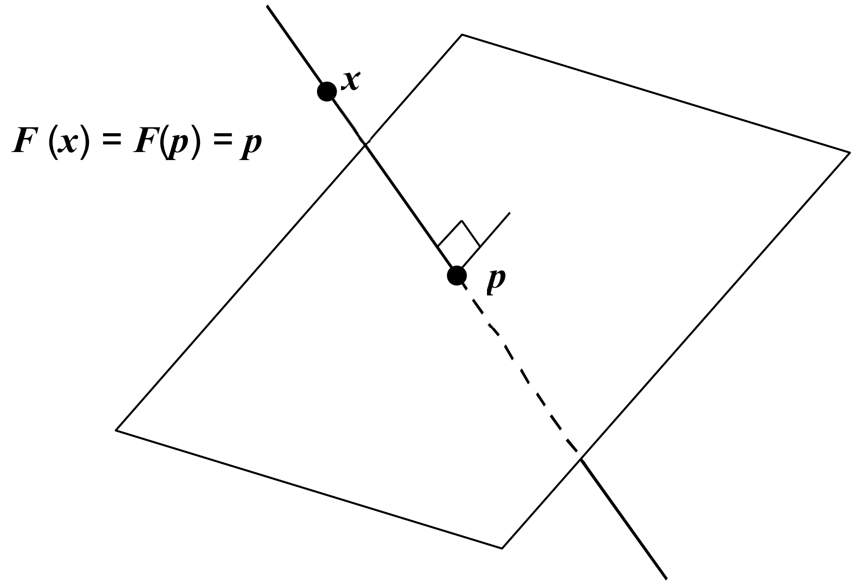

Figure A.4 — Orthographic projection

A.9.2.2 Perspective projection function

Given a plane in $\mathbb{R}^3$ and a point $\boldsymbol{v}$ (the vanishing point) not contained in the plane, the domain of the *perspective projection function* is the set of all points of $\mathbb{R}^3$ in the half space (including the plane) that does not contain the point $\boldsymbol{v}$. Given a point $\boldsymbol{x}$ in the domain, there is one line that passes through both $\boldsymbol{x}$ and $\boldsymbol{v}$. If $\boldsymbol{p}$ is the point at the intersection of the line with the plane, the projection $\boldsymbol{F}$ assigns the value $\boldsymbol{p}$ to $\boldsymbol{x}$. That is $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{p}$. If point $\boldsymbol{q}$ lies in the plane, $\boldsymbol{F}(\boldsymbol{q}) = \boldsymbol{q}$ so that it is a fixed point of the projection. See Figure A.5.

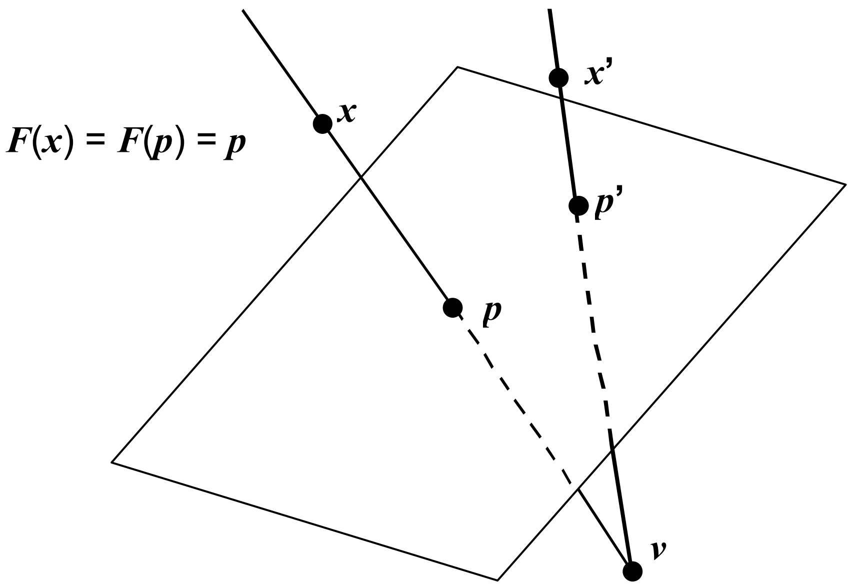

Figure A.5 — Perspective projection

A.9.2.3 Stereographic projection function

Given a plane in $\mathbb{R}^3$ and a point $\boldsymbol{v}$ not contained in the plane, the domain of the *stereographic projection function* is the set of all points of $\mathbb{R}^3$ in the half space on the point $\boldsymbol{v}$ side of (and including) the plane that are closer to the plane than the distance of $\boldsymbol{v}$ to the plane. Given a point $\boldsymbol{x}$ in the domain, there is one line that passes through both $\boldsymbol{x}$ and $\boldsymbol{v}$. If $\boldsymbol{p}$ is the point at the intersection of the line with the plane, the projection $\boldsymbol{F}$ assigns the value $\boldsymbol{p}$ to $\boldsymbol{x}$. That is $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{p}$. If point $\boldsymbol{q}$ lies in the plane, $\boldsymbol{F}(\boldsymbol{q}) = \boldsymbol{q}$ so that it is a fixed point of the projection. See Figure A.6.

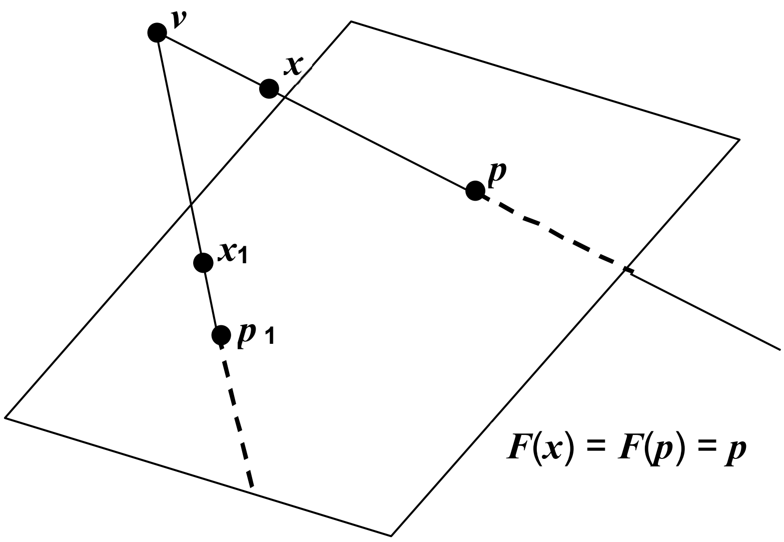

Figure A.6 — Stereographic projection

A.9.3 Cylindrical projection function

Given a cylinder and point $\boldsymbol{v}$ on its axis, a *cylindrical projection function* is defined on the domain $\mathbb{R}^3$excluding the axis points as follows: Given a point $\boldsymbol{x}$ in the domain, there is one ray originating at $\boldsymbol{v}$ that passes through $\boldsymbol{x}$. If $\boldsymbol{p}$ is the point at the intersection of the ray with the cylinder surface, the projection $\boldsymbol{F}$ assigns the value $\boldsymbol{p}$ to $\boldsymbol{x}$. That is $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{p}$. If point $\boldsymbol{q}$ lies on the cylinder surface, $\boldsymbol{F}(\boldsymbol{q}) = \boldsymbol{q}$ so that it is a fixed point of the projection. See Figure A.7.

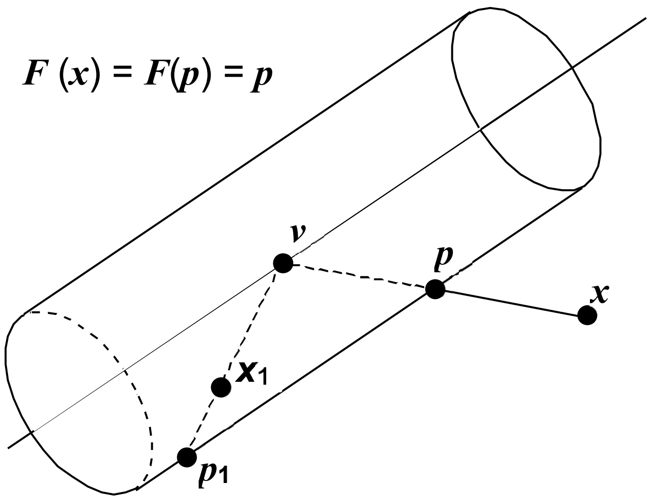

Figure A.7 — Cylindrical projection

A.9.4 Conic projection function

Given a (half) cone and point $\boldsymbol{v}$ on its axis inside the cone, a *conic projection function* projects a point $\boldsymbol{x}$ to the point $\boldsymbol{p}$ where $\boldsymbol{p}$ is the intersection of the cone with the ray from $\boldsymbol{v}$ through $\boldsymbol{x}$. The domain of this projection is the union of all rays originating at $\boldsymbol{v}$ that intersects the cone and excluding the point $\boldsymbol{v}$. See Figure A.8.

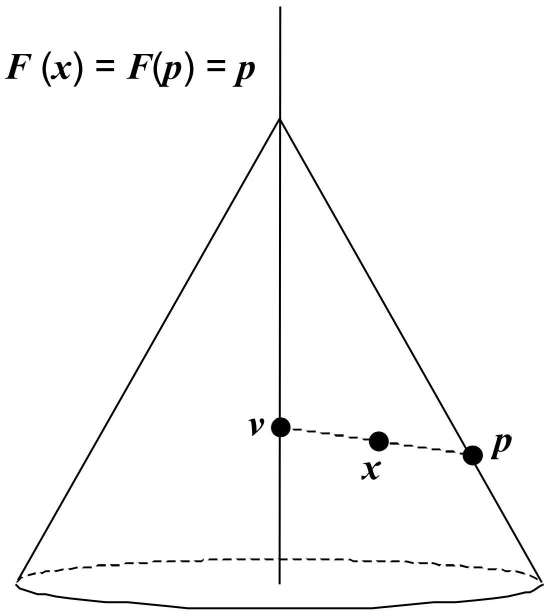

Figure A.8 — Conic projection

A.10 Quaternion Algebra

Let $\boldsymbol{p} = d_0 + d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$ and $q = e_0 + e_1\boldsymbol{i} + e_2\boldsymbol{j} + e_3\boldsymbol{k}$ be two quaternions and let t be a scalar. Quaternion addition and scalar multiplication (in each notational convention) is defined as usual for 4D vector space:

$$\begin{aligned}\boldsymbol{p} + t\boldsymbol{q} &= (d_0 + te_0) + (d_1 + te_1)\boldsymbol{i} + (d_2 + te_2)\boldsymbol{j} + (d_3 + te_3)\boldsymbol{k} && \text{[Hamiltion form]}\\ &= (d_0 + te_0,\ \ \boldsymbol{d} + t\boldsymbol{e}) && \text{[scalar vector form]}\\ &= (d_0 + te_0,\ d_1 + te_1,\ d_2 + te_2,\ d_3 + te_3) && \text{[4-tuple form]}\end{aligned}$$

Assuming associative multiplication, the quaternion axes relationships give the quaternion multiplication rule (in each notational convention):

$$\begin{aligned}\boldsymbol{pq} = &(d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3)\\ &+(d_1e_0 + d_0e_1 + d_2e_3 - d_3e_2)\boldsymbol{i}\\ &+(d_2e_0 + d_0e_2 + d_3e_1 - d_1e_3)\boldsymbol{j}\\ &+(d_3e_0 + d_0e_3 + d_1e_2 - d_2e_1)\boldsymbol{k}\end{aligned} \qquad \text{[Hamiltion form]}$$

$$= \big((d_0e_0 - \boldsymbol{d} \bullet \boldsymbol{e}), (e_0\boldsymbol{d} + d_0\boldsymbol{e} + \boldsymbol{d} \times \boldsymbol{e})\big) \qquad \text{[Scalar vector form]}$$

$$= \begin{bmatrix}(d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3),\\ (d_1e_0 + d_0e_1 + d_2e_3 - d_3e_2),\\ (d_2e_0 + d_0e_2 + d_3e_1 - d_1e_3),\\ (d_3e_0 + d_0e_3 + d_1e_2 - d_2e_1)\end{bmatrix} \qquad \text{[4-tuple form]}$$

Quaternion multiplication is not commutative (the cross product term in the scalar vector form is anti-symmetric). However, the quaternion addition and multiplication operations together form an associative algebra.

The *conjugate* of a quaternion $\boldsymbol{q}$ is defined analogously with complex numbers:

$$\begin{aligned}\boldsymbol{q}^* &= e_0 - e_1\boldsymbol{i} - e_2\boldsymbol{j} - e_3\boldsymbol{k} && \text{[Hamiltion form]}\\ &= (e_0, -\boldsymbol{e}) && \text{[scalar vector form]}\\ &= (e_0, -e_1, -e_2, -e_3) && \text{[4-tuple form]}\end{aligned}$$

The *modulus* of a quaternion is defined as the square root of the product of the quaternion with its conjugate:

$$|\boldsymbol{q}| = \sqrt{\boldsymbol{q}\boldsymbol{q}^*} = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2}, \text{ where:}$$

$$\begin{aligned}\boldsymbol{q}\boldsymbol{q}^* = \boldsymbol{q}^*\boldsymbol{q} &= e_0^2 + e_1^2 + e_2^2 + e_3^2 && \text{[Hamiltion form]}\\ &= (e_0^2 + e_1^2 + e_2^2 + e_3^2,\ \ 0) && \text{[scalar vector form]}\\ &= (e_0^2 + e_1^2 + e_2^2 + e_3^2,\ \ 0,\ \ 0,\ \ 0) && \text{[4-tuple form]}\end{aligned}$$

A quaternion $\boldsymbol{q}$ is a unit quaternion if $|\boldsymbol{q}| = 1$. In that case $\boldsymbol{q}\boldsymbol{q}^* = \boldsymbol{q}^*\boldsymbol{q} = 1$, which implies that for a unit quaternion, its conjugate is its multiplicative inverse $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$. More generally, the inverse of a (non-unit) quaternion $\boldsymbol{p}$ is $\boldsymbol{p}^{-1} = \frac{\boldsymbol{p}^*}{\boldsymbol{p}\boldsymbol{p}^*} = \frac{\mathbf{p}^*}{|\boldsymbol{p}|^2}$.

A.11 Body-fixed rotations in terms of space-fixed axes

In the body-fixed convention (defined in 6.4.2.4), the origin-fixed rotation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ followed by the origin-fixed rotation $\boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle$ is the composite operation $\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ where $\boldsymbol{m}' = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{m})$ is the axis of the second rotation operator after rotation by the first rotation operator.

Define the operator $\boldsymbol{R}$ as $\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle$. Then:

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{m}') &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle(\boldsymbol{m}')\\ &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle\big(\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{m})\big) && \text{substitute for } \boldsymbol{m}'\\ &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ (\boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle)(\boldsymbol{m}) && \text{replace } \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle = \boldsymbol{I}\\ &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle(\boldsymbol{m}) && \text{substitute for } \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle(\boldsymbol{m})\\ &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{m}) = \boldsymbol{m}'.\end{aligned}$$

Thus $\boldsymbol{m}'$ is a unit eigenvector for $\boldsymbol{R}$. Euler's rotation theorem (see 6.4.2.1) then implies that $\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{m}'}\langle\pm\varphi\rangle$ with the sign of φ is to be determined. As can be seen from Rodrigues' rotation formula (Equations 6.2 and 6.6) the limit $\lim\limits_{\theta\to 0} \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle = \boldsymbol{I}$ is the identity operator, taking that limit in the expression for $\boldsymbol{R}(\boldsymbol{m}')$ yields $+\varphi$ as the correct sign of the rotation angle. Hence, $\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle$.

Substitution of this result in the expression $\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ yields:

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle &= (\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle) \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle && \text{substute for } \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle\\ &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ (\boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle) && \text{replace } \boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle = \boldsymbol{I}\\ &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle.\end{aligned}$$

Thus, the body-fixed composite operator $\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ which uses the rotated axis $\boldsymbol{m}'$ is equal to the composite operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle$ that uses the non-rotated axes $\boldsymbol{n}$ and $\boldsymbol{m}$.

A similar result is true for non-origin-fixed rotations. Let $\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle$ denote a rotation through angle θ about the directed axis $\{\boldsymbol{t} + \alpha\boldsymbol{n} | \alpha \in \mathbb{R}\}$ passing through the position vector $\boldsymbol{t}$ and parallel to the unit vector $\boldsymbol{n}$, and let $\boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle$ denote a rotation through angle φ about the directed axis $\{\boldsymbol{u} + \alpha\boldsymbol{m} | \alpha \in \mathbb{R}\}$ passing through the position vector $\boldsymbol{u}$ and parallel to the unit vector $\boldsymbol{m}$.

Consider the consecutive rotations $\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle$ and $\boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle$ in the body-fixed convention. The first rotation $\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle$ does not affect the directed axis $\{\boldsymbol{t}+\alpha\boldsymbol{n} | \alpha \in \mathbb{R}\}$, but does rotate the directed axis $\{\boldsymbol{u}+\alpha\boldsymbol{m} | \alpha \in \mathbb{R}\}$ to a new direction $\boldsymbol{m}' = \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle(\boldsymbol{m})$, and rotates the position vector $\boldsymbol{u}$ to a new position $\boldsymbol{u}' = \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle$. Therefore, the second rotation becomes $\boldsymbol{R}_{\boldsymbol{m}',\boldsymbol{u}'}\langle\varphi\rangle$, yielding the body-fixed composite $\boldsymbol{R}_{\boldsymbol{m}',\boldsymbol{u}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle$. The affine operators $\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle$, $\boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle$ and $\boldsymbol{R}_{\boldsymbol{m}',\boldsymbol{u}'}\langle\varphi\rangle$ may be expressed in terms of origin-fixed rotations. Given an arbitrary vector $\boldsymbol{r}$:

$$\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle(\boldsymbol{r}) = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{r}-\boldsymbol{t}) + \boldsymbol{t}.$$

$$\boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle(\boldsymbol{r}) = \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle(\boldsymbol{r}-\boldsymbol{u}) + \boldsymbol{u}.$$

$$\boldsymbol{R}_{\boldsymbol{m}',\boldsymbol{u}'}\langle\varphi\rangle(\boldsymbol{r}) = \boldsymbol{R}_{\boldsymbol{m}'}\langle\boldsymbol{\varphi}\rangle(\boldsymbol{r}-\boldsymbol{u}') + \boldsymbol{u}'.$$

Substituting, expanding, and simplifying gives:

$$\begin{aligned}
\boldsymbol{R}_{\boldsymbol{m}',\boldsymbol{u}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle(\boldsymbol{r}) &= \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle\big(\boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle(\boldsymbol{r}) - \boldsymbol{u}'\big) + \boldsymbol{u}' \\
&= \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle(\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{r}-\boldsymbol{t}) + \boldsymbol{t} - \boldsymbol{u}') + \boldsymbol{u}' \\
&= \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle\big(\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{r}-\boldsymbol{t}) + \boldsymbol{t} - (\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u}-\boldsymbol{t}) + \boldsymbol{t})\big) + \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u}-\boldsymbol{t}) + \boldsymbol{t} \\
&= \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle\big(\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{r}-\boldsymbol{u})\big) + \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u}-\boldsymbol{t}) + \boldsymbol{t} \\
&= \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle\big(\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{r}-\boldsymbol{u})\big) + \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u}-\boldsymbol{t}) + \boldsymbol{t}
\end{aligned}$$

since $\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle$:

$$\begin{aligned}
&= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle(\boldsymbol{r}-\boldsymbol{u}) + \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u}-\boldsymbol{t}) + \boldsymbol{t} \\
&= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle\big(\boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle(\boldsymbol{r}-\boldsymbol{u}) + (\boldsymbol{u}-\boldsymbol{t})\big) + \boldsymbol{t} \\
&= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle\big((\boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle(\boldsymbol{r}-\boldsymbol{u}) + \boldsymbol{u}) - \boldsymbol{t}\big) + \boldsymbol{t} \\
&= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle\big(\boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle(\boldsymbol{r}) - \boldsymbol{t}\big) + \boldsymbol{t} \\
&= \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle\left(\boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle(\boldsymbol{r})\right) \\
&= \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle(\boldsymbol{r}).
\end{aligned}$$

Hence the general result:

$$\boldsymbol{R}_{\boldsymbol{m}',\boldsymbol{u}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle = \boldsymbol{R}_{\boldsymbol{n},\boldsymbol{t}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m},\boldsymbol{u}}\langle\varphi\rangle.$$

A.12 Rotation and change of basis equivalence

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, a position-vector $\boldsymbol{p}$ is represented by the vector-coordinate $[p_x, p_y, p_z]$ where the coordinate-components satisfy the equation $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$. Since $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ is an orthonormal basis, the solution is unique and is given by the scalars: $p_x = \boldsymbol{p} \bullet \boldsymbol{x}$, $p_y = \boldsymbol{p} \bullet \boldsymbol{y}$, $p_z = \boldsymbol{p} \bullet \boldsymbol{z}$. Thus, for any position-vector $\boldsymbol{p}$ and orthonormal basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$:

$$\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = (\boldsymbol{p} \bullet \boldsymbol{x})\boldsymbol{x} + (\boldsymbol{p} \bullet \boldsymbol{y})\boldsymbol{y} + (\boldsymbol{p} \bullet \boldsymbol{z})\boldsymbol{z}. \tag{A.12.1}$$

The origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ applied to a position-vector $\boldsymbol{p}$ produces a rotated position-vector $\boldsymbol{p}'$. The rotated position-vector $\boldsymbol{p}'$ has vector-coordinates $[p'_x, p'_y, p'_z]$ in frame $\boldsymbol{E}$. The rotation operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$, where the rotation axis $\boldsymbol{n}$ has vector-coordinates $[n_x, n_y, n_z]$ in frame $\boldsymbol{E}$, can be represented as a rotation matrix multiplication $\boldsymbol{p}' = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]\boldsymbol{p}$. The matrix form of $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ is given by Rodrigues' rotation formula (Equation 6.6):

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z\sin\theta & (1-\cos\theta)n_x n_z + n_y\, sin\,\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x\, sin\,\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta & (1-\cos\theta)n_z n_y + n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \tag{A.12.2}$$

The origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ may also be applied to each of the basis vectors of frame $\boldsymbol{E}$, i.e.:

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{x}), \\ \boldsymbol{y}' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{y}), \\ \boldsymbol{z}' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{z}). \end{aligned} \tag{A.12.3}$$

Applying Equation A.12.2 to the basis vectors of the frame $\boldsymbol{E}$ yields the coordinate components of the rotated basis vectors with respect to frame $\boldsymbol{E}$.

$$\boldsymbol{x}' = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]\boldsymbol{x}$$

$$= \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos\theta)n_xn_y - n_z\sin\theta & (1-\cos\theta)n_xn_z + n_y\sin\theta \\ (1-\cos\theta)n_yn_x + n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z - n_x\sin\theta \\ (1-\cos\theta)n_zn_x - n_y\sin\theta & (1-\cos\theta)n_zn_y + n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\boldsymbol{x}' = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x + n_z\sin\theta \\ (1-\cos\theta)n_zn_x - n_y\sin\theta \end{bmatrix}$$

The rotated basis vector $\boldsymbol{x}'$ is represented by the first column of the matrix $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$. Similarly, the rotated basis vectors $\boldsymbol{y}'$ and $\boldsymbol{z}'$ are represented by the second and third columns of the matrix $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$, respectively.

$$\boldsymbol{y}' = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]\boldsymbol{y}$$

$$= \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos\theta)n_xn_y - n_z\sin\theta & (1-\cos\theta)n_xn_z + n_y\sin\theta \\ (1-\cos\theta)n_yn_x + n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z - n_x\sin \\ (1-\cos\theta)n_zn_x - n_y\sin\theta & (1-\cos\theta)n_zn_y + n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\boldsymbol{y}' = \begin{bmatrix} (1-\cos\theta)n_xn_y - n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y + n_x\sin\theta \end{bmatrix}$$

$$\boldsymbol{z}' = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]\boldsymbol{z}$$

$$= \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos)n_xn_y - n_z\sin & (1-\cos\theta)n_xn_z + n_y\sin\theta \\ (1-\cos\theta)n_yn_x + n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z - n_x\sin\theta \\ (1-\cos\theta)n_zn_x - n_y\sin\theta & (1-\cos\theta)n_zn_y + n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\boldsymbol{z}' = \begin{bmatrix} (1-\cos\theta)n_xn_z + n_y\sin\theta \\ (1-\cos\theta)n_yn_z - n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix}$$

The original position-vector $\boldsymbol{p}$ and the rotated position-vector $\boldsymbol{p}'$ each can be represented by vector-coordinates in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and can also be represented by different vector-coordinates with respect to the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$.

The original position-vector $\boldsymbol{p}$ has vector-coordinate $[p_x, p_y, p_z]$ in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and also has vector-coordinate $[p_{x'}, p_{y'}, p_{z'}]$ in terms of the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$.

Similarly, the rotated position-vector $\boldsymbol{p}'$ has vector-coordinate $[p'_x, p'_y, p'_z]$ in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and also has vector-coordinate $[p'_{x'}, p'_{y'}, p'_{z'}]$ in terms of the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$.

Because $\boldsymbol{p}$ is a position-vector, the sum of the products of each of its coordinate-components with the corresponding basis vectors is invariant. Thus:

$$\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = p_{x'}\boldsymbol{x}' + p_{y'}\boldsymbol{y}' + p_{z'}\boldsymbol{z}'.$$

Similarly, because $\boldsymbol{p}'$ is a position-vector:

$$\boldsymbol{p}' = p'_x\boldsymbol{x} + p'_y\boldsymbol{y} + p'_z\boldsymbol{z} = p'_{x'}\boldsymbol{x}' + p'_{y'}\boldsymbol{y}' + p'_{z'}\boldsymbol{z}'.$$

Given $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$, then

$$\begin{aligned} \boldsymbol{p}' &= \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{p}) \\ &= \boldsymbol{R_n}\langle\theta\rangle\left(p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}\right) \\ &= p_x\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{x}) + p_y\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{y}) + p_z\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{z}) \\ &= p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'. \end{aligned} \tag{A.12.4}$$

Thus, the coordinate-components of $\boldsymbol{p}'$ with respect to the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$ have the same values as the coordinate-components of $\boldsymbol{p}$ with respect to the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$.

Using Equation A.12.1, and substituting the expression in Equation A.12.4 for $\boldsymbol{p}'$, the vector-coordinate components of $\boldsymbol{p}'$ are:

$$p'_x = \boldsymbol{p}' \bullet \boldsymbol{x} = \left(p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'\right) \bullet \boldsymbol{x} = p_x(\boldsymbol{x}' \bullet \boldsymbol{x}) + p_y(\boldsymbol{y}' \bullet \boldsymbol{x}) + p_z(\boldsymbol{z}' \bullet \boldsymbol{x}),$$

$$p'_y = \boldsymbol{p}' \bullet \boldsymbol{y} = \left(p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'\right) \bullet \boldsymbol{y} = p_x(\boldsymbol{x}' \bullet \boldsymbol{y}) + p_y(\boldsymbol{y}' \bullet \boldsymbol{y}) + p_z(\boldsymbol{z}' \bullet \boldsymbol{y}),$$

$$p'_z = \boldsymbol{p}' \bullet \boldsymbol{z} = \left(p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'\right) \bullet \boldsymbol{z} = p_x(\boldsymbol{x}' \bullet \boldsymbol{z}) + p_y(\boldsymbol{y}' \bullet \boldsymbol{z}) + p_z(\boldsymbol{z}' \bullet \boldsymbol{z}).$$

The matrix form of these three equations is:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}' \bullet \boldsymbol{x} & \boldsymbol{y}' \bullet \boldsymbol{x} & \boldsymbol{z}' \bullet \boldsymbol{x} \\ \boldsymbol{x}' \bullet \boldsymbol{y} & \boldsymbol{y}' \bullet \boldsymbol{y} & \boldsymbol{z}' \bullet \boldsymbol{y} \\ \boldsymbol{x}' \bullet \boldsymbol{z} & \boldsymbol{y}' \bullet \boldsymbol{z} & \boldsymbol{z}' \bullet \boldsymbol{z} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \tag{A.12.5}$$

Let $\boldsymbol{F}$ denote the orthonormal frame with basis $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$. As discussed in 6.4.3.3, frame $\boldsymbol{F}$ can be conceptually considered to have been rotated away from frame $\boldsymbol{E}$ by the rotation operator $\boldsymbol{R_n}\langle\theta\rangle$. Thus, an alternate notation for the rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ can be given by $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$. The dot product matrix in Equation A.12.5 is therefore a representation of the rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ with respect to orthonormal frame $\boldsymbol{E}$ expressed in terms of the relationships between the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$.

Thus, the rotation matrix $[\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}]$ is identical to the change of basis matrix $[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]$ (see 6.2.2).

$$[\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}] = \begin{bmatrix} \boldsymbol{x}' \bullet \boldsymbol{x} & \boldsymbol{y}' \bullet \boldsymbol{x} & \boldsymbol{z}' \bullet \boldsymbol{x} \\ \boldsymbol{x}' \bullet \boldsymbol{y} & \boldsymbol{y}' \bullet \boldsymbol{y} & \boldsymbol{z}' \bullet \boldsymbol{y} \\ \boldsymbol{x}' \bullet \boldsymbol{z} & \boldsymbol{y}' \bullet \boldsymbol{z} & \boldsymbol{z}' \bullet \boldsymbol{z} \end{bmatrix} = [\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]$$

Expanding each vector in the matrix $[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]$into its components (see Equation A.12.3), and then evaluating each of the dot products, yields the Rodriquez rotation matrix:

$$[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}] = \begin{bmatrix} \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} \\ \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} \\ \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}] = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos\theta)n_xn_y-n_z\sin\theta & (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta & (1-\cos\theta)n_zn_y+n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$$

Thus, the matrix representations of the rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ and the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ are identical. The operators $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ and $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and their matrix representations can also be used to denote the orientation of the rotated frame $\boldsymbol{F}$ with respect to the original frame $\boldsymbol{E}$.

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