10 Operations

10.1 Introduction

This International Standard specifies operations on SRF coordinates and, in the case of 3D object-spaces, on SRF spatial directions, vectors and orientations. Underlying some of these operations are the similarity transformations relating two ORMs (two SRFs with the same ORM is treated as a special case). Similarity transformations are treated first in 10.3. The general case of changing the coordinate of a position in one SRF to its corresponding coordinate in another SRF is specified in 10.4, followed by important special cases. The specification of a spatial direction, vector or orientation in the context of an SRF is defined, and operations for changing these representations from one SRF to their corresponding representations in another SRF are specified in 10.5.

Euclidean distance in 2D and 3D object-space is specified in 10.6. Geodesic distance and azimuth on the surface of an oblate ellipsoid (or sphere) are specified in 10.7.

10.2 Symbols and terminology

An important category of spatial operations is changing the representation of spatial information in one SRF to the representation in a second SRF. For these SRF operations, the adjective “source” shall be used to refer to the first SRF, and the adjective “target” shall be used to refer to the second SRF.

The symbols in Table 10.1 are used throughout this clause.

Table 10.1 — Symbols

Symbol	Definition
$\mathrm{SRF_S}$	Source spatial reference frame
$\mathrm{SRF_T}$	Target spatial reference frame
V_S	Applicable region of $\mathrm{SRF_S}$
E_S	Extended region of $\mathrm{SRF_S}$
$\mathrm{ORM_S}$	Object reference model of $\mathrm{SRF_S}$
$\mathrm{ORM_R}$	Reference ORM for a given spatial object
$\mathrm{CS_S}$	Spatial coordinate system of $\mathrm{SRF_S}$
$\boldsymbol{c}_\mathrm{S}$	Coordinate of a position in $\mathrm{SRF_S}$
$d_E()$	Euclidean distance
$d_G()$	Geodesic distance
$\vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{S}}$	Origin displacement from frame T to frame S
$\boldsymbol{E}$	Embedded orthonormal frame
$\boldsymbol{G}_\mathrm{S}$	Spatial generating function of $\mathrm{CS_S}$
$Dom(\boldsymbol{G}_\mathrm{S})$	Domain of the generating function $\boldsymbol{G}_\mathrm{S}$
$Rng(\boldsymbol{G}_\mathrm{S})$	Range of the generating function $\boldsymbol{G}_\mathrm{S}$
$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$	Similarity transformation from frame *S* to frame T
$\boldsymbol{I}$	Identity matrix (or operator)
$\boldsymbol{L}$	Localized orthonormal frame
$\boldsymbol{L}_\mathrm{3D}$	3D localization operator

Symbol	Definition
$(\lambda_S, \varphi_S, h_S)$	Geodetic coordinate tuple for a position in SRF$_S$
$\boldsymbol{M}_{T\leftarrow S}$	Rotation matrix from frame *S* to frame T
$\boldsymbol{n}_S$	Direction vector in SRF$_S$
$\sigma_{T\leftarrow S}$	Scale factor from frame *S* to frame T
$\boldsymbol{\Omega}_{T\leftarrow S}$	Change of basis operator from frame *S* to frame T
$\boldsymbol{p}$	Position vector
$\boldsymbol{P}_S$	Mapping equations for SRF$_S$
$\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$	Localization parameters
$\boldsymbol{Q}_S$	Inverse mapping equations for SRF$_S$
$\boldsymbol{R}$	Rotation operator
$\boldsymbol{v}_S$	Vector quantity in SRF$_S$
$\boldsymbol{W}$	World 3x3 transformation matrix
$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S$	Position vector components in SRF$_S$
$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}_T$	Origin displacement vector components in SRF$_T$

10.3 ORM operations

10.3.1 Introduction

The similarity transformation (see 7.3.2) $\boldsymbol{H}_{T\leftarrow S}$ between two object reference models, source ORM$_S$ and target ORM$_T$ underlies the coordinate operations in 10.4. There are two cases, depending on whether ORM$_S$ and ORM$_T$ represent the same object, or represent two different objects.

The case where ORM$_S$ and ORM$_T$ represent the same object is addressed in 10.3.2. Although objects are often represented by only a single object reference model, some objects, such as the Earth, are represented by many different object reference models (see Annex E). Given a set of n object reference models for an object, there are $n(n\text{-}1)$ possible source and target ORM pairs. Instead of specifying all possible similarity transformations among these object reference models, this International Standard reduces the requirement to specifying the reference transformation $\boldsymbol{H}_{R\leftarrow S}$ from each source ORM for the object, ORM$_S$ to the designated reference ORM for the object, ORM$_R$.

The more general case where ORM$_S$ and ORM$_T$ represent two different objects is addressed in 10.3.3. This includes subcases where one or both objects are represented by multiple object reference models, and where ORM$_S$ and/or ORM$_T$ are not the reference object reference models for their respective objects. It also includes subcases with different types of relationships between the two objects (see 8.4).

10.3.2 Relating different ORMs for the same object

If ORM$_S$ and ORM$_T$ are different object reference models that represent the same object, and therefore share the same reference ORM, ORM$_R$, the similarity transformation $\boldsymbol{H}_{T\leftarrow S}$ is the composition of their reference transformations $\boldsymbol{H}_{R\leftarrow S}$ and $\boldsymbol{H}_{T\leftarrow R}$, the inverse of $\boldsymbol{H}_{R\leftarrow T}$ as shown in Figure 10.1. This is the common datum transformation operation.

$$\boldsymbol{H}_{T\leftarrow S} = \boldsymbol{H}_{T\leftarrow R} \circ \boldsymbol{H}_{R\leftarrow S} \tag{10.1}$$

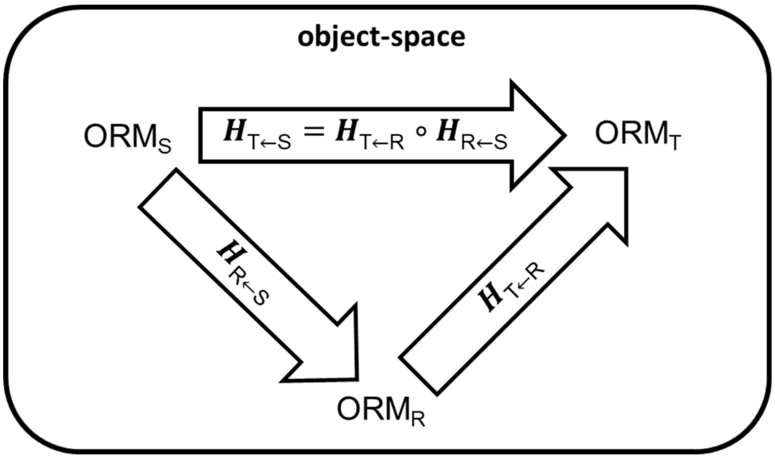

Figure 10.1 — Composed transformations for a single object

If ORM$_S$ is the reference ORM for the object, $\boldsymbol{H}_{R\leftarrow S}$ reduces to the identify $\boldsymbol{I}$. Similarly, if ORM$_T$ is the reference ORM for the object, $\boldsymbol{H}_{T\leftarrow R}$ reduces to the identity $\boldsymbol{I}$.

If ORM$_S$ and ORM$_T$ are identical, the similarity transformation $\boldsymbol{H}_{T\leftarrow S}$ reduces to the identify $\boldsymbol{I}$ (see 10.4.3 and 10.4.4). This subcase includes the relationship between a regional SRF and another SRF used as a reference (see 8.4.2).

If ORM$_S$ is an object-fixed ORM, its reference transformation $\boldsymbol{H}_{R\leftarrow S}$ is a type of similarity transformation. Any 3D or 2D similarity transformation may be represented with the STT ROTATE_SCALE_TRANSLATE in the 3D case or STT ROTATE_SCALE_TRANSLATE_2D in the 2D case. Thus, using the notation of the STT formulation, $\boldsymbol{H}_{R\leftarrow S}$ may be represented as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R = \boldsymbol{H}_{R\leftarrow S}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S\right) \equiv \vec{\Delta}_{R\leftarrow S} + \sigma_{R\leftarrow S}\boldsymbol{M}_{R\leftarrow S}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S \tag{10.2}$$

NOTE For the Earth, the processes by which object reference models are established are based on physical measurements. These measurements are subject to error, and therefore introduce various types of relative distortions between object reference models. The scale factor $\sigma_{R\leftarrow S}$ in Equation 10.2 should equal 1,0 since each ORM is for the same object-space. However, values very close to 1,0 are allowed to account for small distortions (see 7.3.2). The reference transformation $\boldsymbol{H}_{R\leftarrow T}$ from ORM$_T$ to the reference ORM$_R$ is also a similarity transformation.

$\boldsymbol{H}_{T\leftarrow R}$ is also a similarity transformation:

$$\boldsymbol{H}_{T\leftarrow R}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R\right) = \boldsymbol{H}_{R\leftarrow T}^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R\right) = \left({}^{1}\!/_{\sigma_{R\leftarrow T}}\right)\boldsymbol{M}_{R\leftarrow T}^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R - \vec{\Delta}_{R\leftarrow T}\right)$$
$$= \vec{\Delta}_{T\leftarrow R} + \left({}^{1}\!/_{\sigma_{R\leftarrow T}}\right)\boldsymbol{M}_{R\leftarrow T}^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R$$

Because the matrix $\boldsymbol{M}_{R\leftarrow T}$ is a rotation matrix, its transpose $\boldsymbol{M}_{R\leftarrow T}^{T}$ is also its inverse $\boldsymbol{M}_{R\leftarrow T}^{-1}$. The inverse of $\boldsymbol{M}_{R\leftarrow T}$ is also the matrix $\boldsymbol{M}_{T\leftarrow R}$ corresponding to the reverse rotations of ORM$_T$ with respect to ORM$_R$. In particular:

$$\boldsymbol{M}_{T\leftarrow R} = \boldsymbol{M}_{R\leftarrow T}^{-1} = \boldsymbol{M}_{R\leftarrow T}^{T}$$

and

$$\boldsymbol{H}_{T\leftarrow R}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R\right) = \vec{\Delta}_{T\leftarrow R} + \left({}^{1}\!/_{\sigma_{R\leftarrow T}}\right)\boldsymbol{M}_{T\leftarrow R}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_R.$$

The composite operation $\boldsymbol{H}_{T\leftarrow S} = \boldsymbol{H}_{T\leftarrow R} \circ \boldsymbol{H}_{R\leftarrow S}$ reduces to:

$$\boldsymbol{H}_{T\leftarrow S}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S\right) = \boldsymbol{H}_{T\leftarrow R} \circ \boldsymbol{H}_{R\leftarrow S}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S\right) = \vec{\Delta}_{T\leftarrow S} + \left({}^{\sigma_{R\leftarrow S}}\!/_{\sigma_{R\leftarrow T}}\right)\boldsymbol{M}_{T\leftarrow S}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S \tag{10.3}$$

where:

$$\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{S}} = \boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{R}}\,\boldsymbol{M}_{\mathrm{R}\leftarrow\mathrm{S}},\ \text{and}\ \vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{S}} = \vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{R}} + \left({}^{1}\!/_{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\right)\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{R}}\,\vec{\Delta}_{\mathrm{R}\leftarrow\mathrm{S}}.$$

If the rotations $\boldsymbol{M}_{\mathrm{R}\leftarrow\mathrm{S}}$ and $\boldsymbol{M}_{\mathrm{R}\leftarrow\mathrm{T}}$ are equal, then $\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{S}}$ is the identity matrix, and if $\sigma_{\mathrm{R}\leftarrow\mathrm{S}} = \sigma_{\mathrm{R}\leftarrow\mathrm{T}}$, $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ simplifies to a translation of the origin:

$$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{S}}\right) = \vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{S}} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathrm{S}}.$$

Equation 10.1 and Figure 10.1 also apply to the 2D case.

If the source ORM$_{\mathrm{S}}$ is a time-dependent ORM for a spatial object, ORM$_{\mathrm{S}}(t)$ shall denote the source ORM$_{\mathrm{S}}$ at time t, and $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}(t)$ shall denote the similarity transformation from ORM$_{\mathrm{S}}(t)$ to the object-fixed reference ORM$_{\mathrm{R}}$. For a fixed value of time t_0, Equation 10.1 and Figure 10.1 generalize to $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}(t_0) = \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}} \circ \boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}(t_0)$. The generalization to a time-dependent target ORM$_{\mathrm{T}}(t)$ is $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}(t_0) = \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}(t_0) \circ \boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$. The generalization when both ORMs are time-dependent at time t_0 is $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}(t_0) = \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}(t_0) \circ \boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}(t_0)$.

EXAMPLE ORM$_{\mathrm{S}}(t)$ is the ORM EARTH_INERTIAL_J2000r0 at time t. ORM$_{\mathrm{R}}$ is the Earth reference ORM WGS_1984. Because ORM$_{\mathrm{S}}(t)$ and ORM$_{\mathrm{R}}$ share the same embedding origin, the $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}(t)$ transformation is a (rotation) matrix multiplication operation (without translation). The matrix coefficients for selected values of t account for polar motion, Earth rotation, nutation, and precession. Predicted values for these coefficients are computed and updated weekly by the International Earth Rotation and Reference Systems Service (IERS) [IERS36]. See 7.5 for other examples of dynamic ORM reference transformations.

10.3.3 Relating ORMs for different objects

If ORM$_{\mathrm{S}}$ and ORM$_{\mathrm{T}}$ are different object reference models that represent two different objects, a source object $\boldsymbol{S}$ and a target object $\boldsymbol{T}$, the similarity transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ is the composition of the reference transformation for ORM$_{\mathrm{S}}$, $\boldsymbol{H}_{\mathrm{R_S}\leftarrow\mathrm{S}}$, the similarity transformation between the reference object reference models of the two objects, $\boldsymbol{H}_{\mathrm{R_T}\leftarrow\mathrm{R_S}}$, and the inverse reference transformation for ORM$_{\mathrm{T}}$, $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R_T}}$, as shown in Figure 10.2.

$$\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}} = \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R_T}} \circ \boldsymbol{H}_{\mathrm{R_T}\leftarrow\mathrm{R_S}} \circ \boldsymbol{H}_{\mathrm{R_S}\leftarrow\mathrm{S}} \tag{10.4}$$

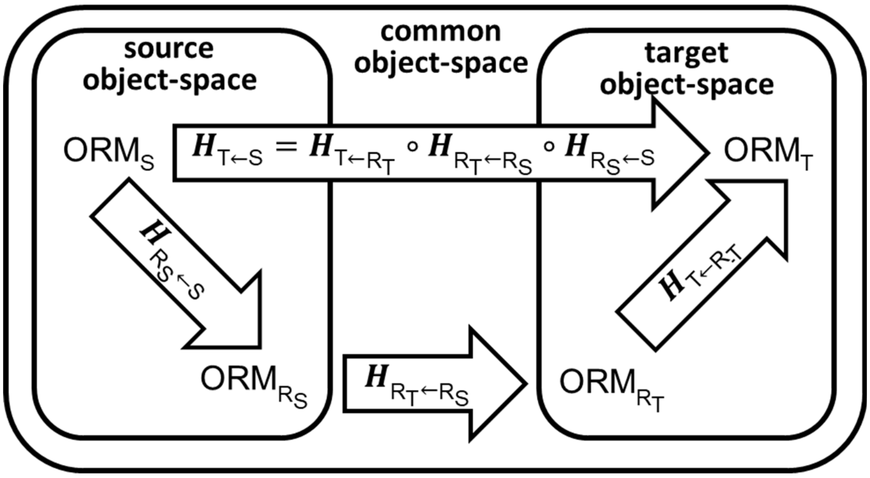

Figure 10.2 — Composed transformations for two different objects

The similarity transformations $\boldsymbol{H}_{\mathrm{R_S}\leftarrow\mathrm{S}}$ and $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R_T}}$ are the same as the corresponding transformations $\boldsymbol{H}_{\mathrm{R}\leftarrow\mathrm{S}}$ and $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R}}$ in 10.3.2. If ORM$_{\mathrm{S}}$ is the reference ORM for the source object, $\boldsymbol{H}_{\mathrm{R_S}\leftarrow\mathrm{S}}$ reduces to the identity $\boldsymbol{I}$. Similarly, if ORM$_{\mathrm{T}}$ is the reference ORM for the target object, $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{R_T}}$ reduces to the identity $\boldsymbol{I}$.

Given that the two objects are fixed with respect to each other, the similarity transformation between their reference object reference models, $\boldsymbol{H}_{\mathrm{R_T}\leftarrow\mathrm{R_S}}$, depends on the relationship between the objects and their object-spaces. If one of the objects represents an assembly that includes the other object as a component, the object-space of the component object can be considered to be nested within the object-space of the assembly object, as discussed in 8.4.2. In that case, the common object-space shown in Figure 10.2 represents the assembly object-space, and the similarity transformation $\boldsymbol{H}_{\mathrm{R_T} \leftarrow \mathrm{R_S}}$ can be derived from the known displacement and orientation relationships between the object reference models of the component and assembly objects.

If the two objects are independent of each other, it still may be possible to derive the similarity transformation $\boldsymbol{H}_{\mathrm{R_T} \leftarrow \mathrm{R_S}}$ if the displacement and orientation relationships between the two objects can be determined. As discussed in 8.4.2. it may be possible to consider one object as operating within the object-space of the other object. In that case, the ORM of the second object provides a reference for the ORM of the first object. Alternatively, it may be possible to consider both objects as operating within the object-space of a third object. In that case, the ORM of the third object provides a reference for both objects. In these last two cases, the common object-space shown in Figure 10.2 represents the object-space of the reference object,

If any ORM involved in the transformation $\boldsymbol{H}_{\mathrm{T} \leftarrow \mathrm{S}}$ is time-dependent, ORM(t) shall denote that ORM at time t. Any similarity transformations involving that ORM are also time-dependent, and shall be denoted $\boldsymbol{H}_{\mathrm{T} \leftarrow \mathrm{S}}(t)$. If the relationship between the object reference models can be determined at a fixed value of time t_0, the similarity transformations generalize in the manner described in 10.3.2.

EXAMPLE ORM$_\mathrm{S}$ is the reference ORM for the space shuttle (as source object $\boldsymbol{S}$). ORM$_\mathrm{T}$ is the reference ORM WGS_1984 for the Earth (as spatial object $\boldsymbol{T}$). When in orbit, the object-space of the space shuttle can be considered to be nested within the object-space of the Earth. At time t, the position and orientation of ORM$_\mathrm{S}$ with respect to ORM$_\mathrm{T}$ are known. $\boldsymbol{H}_{\mathrm{T} \leftarrow \mathrm{S}}(t)$ can be determined and used to transform positions with respect to ORM$_\mathrm{S}$ to positions with respect to ORM$_\mathrm{T}$.

10.4 Position operations

10.4.1 Introduction

Given a coordinate $\boldsymbol{c}_\mathrm{S}$ representing a position in a source SRF, SRF$_\mathrm{S}$, the operation[25] that computes the corresponding coordinate $\boldsymbol{c}_\mathrm{T}$ of that position in a given target SRF, SRF$_\mathrm{T}$ is termed a change of SRF operation. This is a generalization of the change of basis operation defined in 6.2.

The general case of the change of SRF operation is addressed in 10.4.2. The general case depends on the existence of a similarity transformation $\boldsymbol{H}_{\mathrm{T} \leftarrow \mathrm{S}}$ (see 10.3) from the embedded frame determined by ORM$_\mathrm{S}$, the ORM associated with SRF$_\mathrm{S}$, to the embedded frame determined by ORM$_\mathrm{T}$, the ORM associated with SRF$_\mathrm{T}$. The general case also depends on CS$_\mathrm{S}$, the spatial coordinate system associated with SRF$_\mathrm{S}$, and CS$_\mathrm{T}$, the spatial coordinate system associated with SRF$_\mathrm{T}$.

Special cases allow for simplifications that result in computational short cuts to the general case. The case of matched normal embeddings is addressed in 10.4.3. Further specializations arise from combinations of specific coordinate-systems. Subclause 10.4.4 treats combinations of celestiodetic with a map projection.

Cases where CS$_\mathrm{S}$ and CS$_\mathrm{T}$ are based on the same abstract coordinate system, but ORM$_\mathrm{S}$ and ORM$_\mathrm{T}$ differ[26] do not generally produce computational simplifications. However, SRF$_\mathrm{S}$ and SRF$_\mathrm{T}$ are based on the LOCOCENTRIC_EUCLIDEAN_3D CS, a simplification is possible. This simplification is presented in 10.4.5. This simplification is important for operations on directions, vector quantities, and orientations (see 10.5).

Another important special case occurs when the source object space is an abstract 3D object space. This special case is treated in 10.4.6.

10.4.2 General case

In the general case of the change of SRF operation, the source and target SRFs, SRF$_\mathrm{S}$ and SRF$_\mathrm{T}$, are each based on a spatial coordinate system, CS$_\mathrm{S}$ and CS$_\mathrm{T}$. SRF$_\mathrm{S}$ and SRF$_\mathrm{T}$ are also each based on an object reference

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[25] ISO 19111 defines this case as a coordinate operation.

[26] ISO 19111 defines this case as a coordinate transformation.

model, ORMs and ORMT. SRFs and SRFT can be associated with different objects or with the same object. If SRFs and SRFT are associated with the same object, they can be based on different object reference models for that object, or on the same ORM.

Given two object-fixed SRFs, SRFs and SRFT, and a point in an object-space $\boldsymbol{p}$ that is within the applicable regions of both SRFs, the most general form of the change of SRF operation is:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{G}_{\mathrm{T}}^{-1} \circ \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}} \circ \boldsymbol{G}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}}) \quad (10.5)$$

where $\boldsymbol{c}_{\mathrm{S}}$ denotes the coordinate of $\boldsymbol{p}$ in SRFs, and $\boldsymbol{c}_{\mathrm{T}}$ denotes the coordinate of $\boldsymbol{p}$ in SRFT. $\boldsymbol{G}_{\mathrm{S}}$ is the spatial generating function for CSs. $\boldsymbol{G}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}})$ is the position vector $\boldsymbol{p}$ expressed in the embedded frame determined by ORMs. $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ is the similarity transformation that transforms $\boldsymbol{p}$ from the embedded frame determined by ORMs to the embedded frame determined by ORMT. The inverse of the spatial generating function $\boldsymbol{G}_{\mathrm{T}}$, operating on $\boldsymbol{p}$ expressed in terms of the embedded frame determined by ORMT, returns $\boldsymbol{c}_{\mathrm{T}}$. The composition of these operations is illustrated in Figure 10.3. CS generating and inverse generating functions are specified in Clause 5. Similarity transformations are specified in Clause 7.

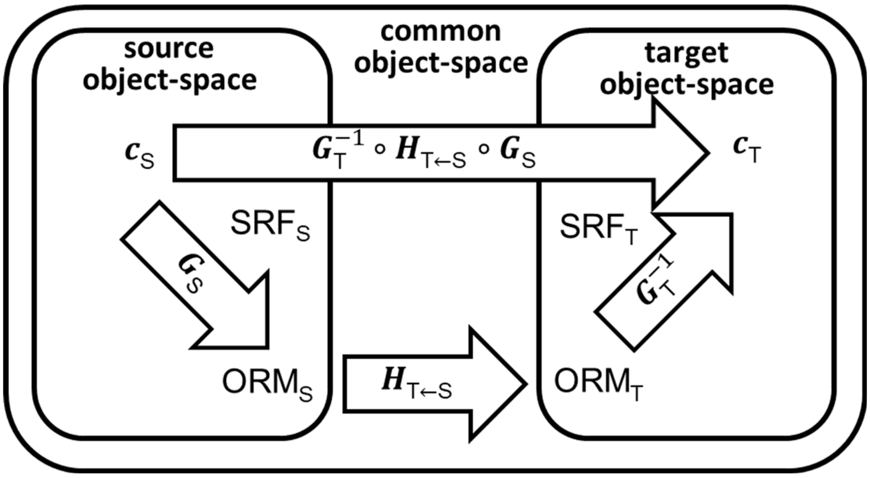

Figure 10.3 — Change of SRF operation – applied to coordinates

Equation 10.5 is only defined for a value of $\boldsymbol{c}_{\mathrm{S}}$ in the CSs domain if its corresponding position belongs to the CST range (the range of a generating function is the domain of its inverse generating function). If $Dom(\boldsymbol{G}_{\mathrm{S}})$ is the domain of the generating function $\boldsymbol{G}_{\mathrm{S}}$, $Rng(\boldsymbol{G}_{\mathrm{S}})$ is the range of the generating function $\boldsymbol{G}_{\mathrm{S}}$, and $Rng(\boldsymbol{G}_{\mathrm{T}})$ is the range of the generating function $\boldsymbol{G}_{\mathrm{T}}$, Equation 10.5 is only defined for $\boldsymbol{c}_{\mathrm{S}}$ in the set:

$$\boldsymbol{G}_{\mathrm{S}}^{-1}\left(Rng(\boldsymbol{G}_{\mathrm{S}}) \cap \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}^{-1}\big(Rng(\boldsymbol{G}_{\mathrm{T}})\big)\right) \equiv \left\{\boldsymbol{c}_{\mathrm{S}} \text{ in } Dom(\boldsymbol{G}_{\mathrm{S}}) \middle| \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}\big(\boldsymbol{G}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}})\big) \text{ in } Rng(\boldsymbol{G}_{\mathrm{T}})\right\} \quad (10.6)$$

If $\boldsymbol{c}_{\mathrm{S}}$ does not belong to this set, it is invalid for the operation in Equation 10.5.

EXAMPLE SRFs is SRF GEOCENTRIC_WGS_1984 and SRFT is an instance of SRF template MERCATOR, with ORM WGS_1984. For any $\boldsymbol{c}_{\mathrm{S}}$ that is on the *z*-axis of SRFs, Equation 10.5 is not defined and is thus invalid, because the *z*-axis is not contained in the range of SRF template MERCATOR and, thus, it is not contained in the set in Equation 10.6.

SRFT may optionally specify an applicable region V_{T}, and may optionally also specify an extended region E_{T} (see 8.3.2.4). If $Dom(\boldsymbol{G}_{\mathrm{T}})$ is the domain of the generating function $\boldsymbol{G}_{\mathrm{T}}$, then $V_{\mathrm{T}} \subseteq E_{\mathrm{T}} \subseteq Dom(\boldsymbol{G}_{\mathrm{T}})$. If $\boldsymbol{c}_{\mathrm{T}}$ is computed using Equation 10.5, $\boldsymbol{c}_{\mathrm{T}}$ is either within the applicable region ($\boldsymbol{c}_{\mathrm{T}}$ is in V_{T}), or $\boldsymbol{c}_{\mathrm{T}}$ is within the extended region but not within the applicable region ($\boldsymbol{c}_{\mathrm{T}}$ is in $V_{\mathrm{T}}/E_{\mathrm{T}}$), or $\boldsymbol{c}_{\mathrm{T}}$ is within the CS domain but not within the extended region ($\boldsymbol{c}_{\mathrm{T}}$ is in $Dom(\boldsymbol{G}_{\mathrm{T}})\backslash E_{\mathrm{T}}$).

In applications that functionally conform to an SRM profile, the domain of an SRF operation is restricted to the accuracy domain of the SRF as specified by that profile (see Clause 12).

Equation 10.5 depends on the existence of a similarity transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ from the embedded frame determined by ORMs to the embedded frame determined by ORMT. If ORMs and ORMT represent the same object, $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ is as defined in 10.3.2. If ORMs and ORMT represent different objects, $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ is as defined in 10.3.3. The simplifications

If SRF$_S$ and SRF$_T$ are two celestiodetic SRFs with different object reference models for the same spatial object, Equation 10.5 transforms the coordinate $\boldsymbol{c}_S = (\lambda_S, \varphi_S, h_S)$ with respect to one oblate ellipsoid to $\boldsymbol{c}_T = (\lambda_T, \varphi_T, h_T)$ with respect to the other oblate ellipsoid. A transformation between two celestiodetic SRFs for the spatial object Earth is known as a *horizontal datum shift*.

NOTE A number of numerical approximations developed to implement horizontal datum shift have been published. Under the assumption of zero rotations and no scale differences, a widely used approximation[27] to directly transform $\boldsymbol{c}_S = (\lambda_S, \varphi_S, h_S)$ to $\boldsymbol{c}_T = (\lambda_T, \varphi_T, h_T)$ is the *standard Molodensky transformation* formula (see [NGA36]).

In the case of a time-dependent relationships between ORM$_S$ and ORM$_T$, Equation 10.5 generalizes to:

$$\boldsymbol{c}_T(t) = \boldsymbol{G}_T^{-1} \circ \boldsymbol{H}_{T \leftarrow S}(t) \circ \boldsymbol{G}_S(\boldsymbol{c}_S)$$

The time-dependent similarity transformation $\boldsymbol{H}_{T \leftarrow S}(t)$ is as discussed in 10.3.2 and 10.3.3, depending on whether ORM$_S$ and ORM$_T$ represent the same object or two different objects.

10.4.3 Matched normal embeddings

In this special case of the change of SRF operation, the source and target SRFs share the same ORM, or, more generally, the reference transformations of ORM$_S$ and ORM$_T$ are equivalent (*i.e.*, matched normal embeddings), and therefore $\boldsymbol{H}_{T \leftarrow S}$ is the identity transformation. Consequently, Equation 10.5 simplifies to:

$$\boldsymbol{c}_T = \boldsymbol{G}_T^{-1} \circ \boldsymbol{G}_S(\boldsymbol{c}_S) \quad \text{for all } \boldsymbol{c}_S \text{ in the set: } \left\{\boldsymbol{G}_S^{-1}\left(Rng(\boldsymbol{G}_S) \cap Rng(\boldsymbol{G}_T)\right)\right\}. \qquad (10.7)$$

EXAMPLE 1 If SRF$_S$ is a celestiodetic SRF and SRF$_T$ is the celestiocentric SRF for the same ORM, then since the CS of the celestiocentric SRF is Euclidean_3D for which the $\boldsymbol{G}_T^{-1}$ is the identity, Equation 10.7 reduces to the geodetic generating function: $\boldsymbol{c}_T = \boldsymbol{G}_S(\boldsymbol{c}_S)$.

If SRF$_T$ is a 3D SRF that has ellipsoidal height designated as the vertical coordinate-component of the SRF (see 8.4.3), and SRF$_S$ is the induced zero height surface SRF, the *promotion operation* converts a surface coordinate $\boldsymbol{c}_S$ in SRF$_S$ to a 3D coordinate in SRF$_T$ by setting the 1st and 2nd coordinate-components of $\boldsymbol{c}_T$ to the 1st and 2nd coordinate-components of $\boldsymbol{c}_S$ and setting the 3rd coordinate-component, ellipsoidal height, to 0. Coordinate promotion is a special case of Equation 10.7. Applicable spatial reference frames include those based on SRF templates CELESTIODETIC, PLANETODETIC, and all map projection SRF templates.

EXAMPLE 2 If SRF$_S$ is an induced zero height surface celestiodetic SRF and SRF$_T$ is the 3D celestiodetic SRF for the same ORM, Equation 10.7 promotes $\boldsymbol{c}_S = (\lambda, \varphi)$ from a coordinate of CS type surface to $\boldsymbol{c}_T = (\lambda, \varphi, 0)$ a coordinate of CS type 3D.

If SRF$_S$ is a 3D SRF that has ellipsoidal height designated as the vertical coordinate-component of the SRF (see 8.4.3), and SRF$_T$ is the induced zero height surface SRF, the *truncation operation* converts a 3D coordinate $\boldsymbol{c}_S$ in SRF$_S$ to a surface coordinate $\boldsymbol{c}_T$, by setting the 1st and 2nd coordinate-components of $\boldsymbol{c}_T$ to the 1st and 2nd coordinate-components of $\boldsymbol{c}_S$. The point in object-space corresponding to $\boldsymbol{c}_S$ and the point in object-space corresponding to $\boldsymbol{c}_T$ are not the same point unless the height coordinate-component $h = 0$. Truncation, therefore, does not generally preserve location.

EXAMPLE 3 If SRF$_S$ is a celestiodetic 3D SRF, the (induced) zero height surface SRF$_T$ is the surface celestiodetic SRF for the same ORM. The truncation operation associates $\boldsymbol{c}_T = (\lambda, \varphi)$ to $\boldsymbol{c}_S = (\lambda, \varphi, 0)$.

EXAMPLE 4 SRF$_S$ is a celestiodetic 3D SRF based on ORM SIRGAS_2000 (Table D.2). SRF$_T$ is a celestiodetic 3D SRF based on ORM WGS_84, which is the reference ORM for Earth. The reference transformation for SIRGAS_2000 (Table E.6) is the identity transformation, thus the ORM embedded frames match and Equation 10.7 applies. However, the ellipsoid RDs for these two ORM have differing minor semi-axis values b. Thus, the generating functions for these SRFs, while both

[27] Historically it was thought that these approximations would require less computation than direct conversion. The perceived computational advantage may have been overcome by technology advances. New efficient algorithms for converting celestiocentric coordinates to celestiodetic coordinates have been developed that result in appreciably faster transformations without the attendant loss of accuracy.

celestiodetic 3D, have differing values at non-zero latitudes. Consequently $\boldsymbol{G}_{\mathrm{T}}^{-1} \circ \boldsymbol{G}_{\mathrm{S}}$ in Equation 10.6 will not equal the identity function. Furthermore, the range of SRF$_{\mathrm{S}}$ is smaller than the range of SRF$_{\mathrm{T}}$.

10.4.4 Matched normal embeddings and map projection SRFs

In this special case of the change of SRF operation for map projection spatial reference frames, the source and target spatial reference frames share the same ORM, or, more generally, the reference transformations of ORM$_{\mathrm{S}}$ and ORM$_{\mathrm{T}}$ determine the same embedded frame (*i.e.*, matched normal embeddings), and therefore $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ is the identity transformation.

The spatial CS generating function $\boldsymbol{G}_{\mathrm{MP}}$ for an augmented map projection SRF is implicitly defined (see 5.3.7.2 and 5.4.2) by the composition of the spatial generating function, $\boldsymbol{G}_{\mathrm{GD}}$, for the geodetic 3D CS with the inverse mapping equation $\boldsymbol{Q} \equiv (Q_1, Q_2, h)$ as:

$$\boldsymbol{G}_{\mathrm{MP}} = \boldsymbol{G}_{\mathrm{GD}} \circ \boldsymbol{Q}.$$

If SRF$_{\mathrm{S}}$ and SRF$_{\mathrm{T}}$ are map projection spatial reference frames for the same object, and the reference transformations of ORM$_{\mathrm{S}}$ and ORM$_{\mathrm{T}}$ are equivalent, Equation 10.7 becomes:

$$\begin{aligned} \boldsymbol{c}_{\mathrm{T}} &= (\boldsymbol{G}_{\mathrm{GD,T}} \circ \boldsymbol{Q}_{\mathrm{T}})^{-1} \circ (\boldsymbol{G}_{\mathrm{GD,S}} \circ \boldsymbol{Q}_{\mathrm{S}})(\boldsymbol{c}_{\mathrm{S}}) \\ &= \boldsymbol{P}_{\mathrm{T}} \circ \boldsymbol{G}_{\mathrm{GD,T}}^{-1} \circ \boldsymbol{G}_{\mathrm{GD,S}} \circ \boldsymbol{Q}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}}) \end{aligned} \tag{10.8}$$

where:

$\boldsymbol{Q}_{\mathrm{S}}$: inverse mapping equations for SRF$_{\mathrm{S}}$,

$\boldsymbol{G}_{\mathrm{GD,S}}$: spatial generating function for the geodetic 3D CS for SRF$_{\mathrm{S}}$,

$\boldsymbol{Q}_{\mathrm{T}}$: inverse mapping equations for SRF$_{\mathrm{T}}$ (the inverse of $\boldsymbol{P}_{\mathrm{T}}$)

$\boldsymbol{P}_{\mathrm{T}}$: mapping equations for SRF$_{\mathrm{T}}$, and

$\boldsymbol{G}_{\mathrm{GD,T}}$: spatial generating function for the geodetic 3D CS for SRF$_{\mathrm{T}}$,

Furthermore, if ORM$_{\mathrm{S}}$ = ORM$_{\mathrm{T}}$, $\boldsymbol{G}_{\mathrm{GD,S}} = \boldsymbol{G}_{\mathrm{GD,T}}$ and Equation 10.8 simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{P}_{\mathrm{T}} \circ \boldsymbol{Q}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}}). \tag{10.9}$$

If SRF$_{\mathrm{T}}$ is a celestiodetic SRF, SRF$_{\mathrm{S}}$ is an augmented map projection SRF, and ORM$_{\mathrm{T}}$ = ORM$_{\mathrm{S}}$, Equation 10.7 simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \mathbf{Q}_{\mathrm{S}}(\boldsymbol{c}_{\mathrm{S}}).$$

Similarly, if SRF$_{\mathrm{S}}$ is a celestiodetic SRF, SRF$_{\mathrm{T}}$ is an augmented map projection SRF, and ORM$_{\mathrm{T}}$ = ORM$_{\mathrm{S}}$, Equation 10.7 simplifies to:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{P}_{\mathrm{T}}(\boldsymbol{c}_{\mathrm{S}}).$$

10.4.5 Cartesian 3D SRFs

In this special case of the change of SRF operation both the source and target SRFs (SRF$_{\mathrm{S}}$ and SRF$_{\mathrm{T}}$) are instances of the LOCOCENTRIC_EUCLIDEAN_3D SRF template (Table 8.11). This special case is important for the treatment of directions, vectors, and orientations (see 10.5). This SRFT requires localization parameter vectors $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ in the embedded frame $\boldsymbol{E}$ determined by the associated ORM. In terms of these parameters the spatial generating function, $\boldsymbol{G}_{\mathrm{LE3D}}$, is in the form of an affine transformation and thus allows the change of SRF operation to be explicitly expressed in affine transformation form (Equation 10.10) as well. The affine form of $\boldsymbol{G}_{\mathrm{LE3D}}$ operating on the coordinate (u, v, w) of a position $\boldsymbol{p}$ in the localized frame $\boldsymbol{L}$ is:

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{G}_{\mathrm{LE3D}}\big((u, v, w)\big) = \boldsymbol{L}_{\mathrm{3D}} \circ \boldsymbol{G}_{\mathrm{E3D}}\big((u, v, w)\big) \\ &= \boldsymbol{q} + u\boldsymbol{r} + v\boldsymbol{s} + w\boldsymbol{t} \\ &= \boldsymbol{q} + u \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + v \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + w \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \end{aligned}$$

$$= \boldsymbol{q} + \begin{bmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \boldsymbol{q} + \boldsymbol{\Omega}_{\boldsymbol{E} \leftarrow \boldsymbol{L}} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where:

$$\boldsymbol{\Omega}_{\boldsymbol{E} \leftarrow \boldsymbol{L}} = \begin{bmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \bullet \boldsymbol{x} & \boldsymbol{s} \bullet \boldsymbol{x} & \boldsymbol{t} \bullet \boldsymbol{x} \\ \boldsymbol{r} \bullet \boldsymbol{y} & \boldsymbol{s} \bullet \boldsymbol{y} & \boldsymbol{t} \bullet \boldsymbol{y} \\ \boldsymbol{r} \bullet \boldsymbol{z} & \boldsymbol{s} \bullet \boldsymbol{z} & \boldsymbol{t} \bullet \boldsymbol{z} \end{bmatrix},$$

$\boldsymbol{r}, \boldsymbol{s}$ and $\boldsymbol{t} = (\boldsymbol{r} \times \boldsymbol{s})$ are the basis vectors of the localized frame $\boldsymbol{L}$, and
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are the basis vectors of the embedded frame $\boldsymbol{E}$.

The spatial generating function $\boldsymbol{G}_{\text{LE3D}}$ maps a coordinate tuple in the domain of the localized frame of the SRF to the corresponding position $\boldsymbol{p}$ in terms of the embedded frame $\boldsymbol{E}$ determined by the ORM of the SRF. The coordinate tuple (u, v, w) corresponds to the column vector $[u \quad v \quad w]^{\mathrm{T}}$, for $\boldsymbol{p}$ in the localized frame $\boldsymbol{L}$ specified by the parameters $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$.

The inverse generating function can be similarly expressed as:

$$\boldsymbol{G}_{\text{LE3D}}^{-1}(\boldsymbol{p}) = (u, v, w)$$

where $[u \quad v \quad v]^{\mathrm{T}} = \boldsymbol{\Omega}_{\boldsymbol{L} \leftarrow \boldsymbol{E}}(\boldsymbol{p} - \boldsymbol{q})$ and $\boldsymbol{\Omega}_{\boldsymbol{L} \leftarrow \boldsymbol{E}} = \boldsymbol{\Omega}_{\boldsymbol{E} \leftarrow \boldsymbol{L}}^{-1} = \boldsymbol{\Omega}_{\boldsymbol{E} \leftarrow \boldsymbol{L}}^{\mathrm{T}}$

The change of basis operation $\boldsymbol{\Omega}_{\boldsymbol{L} \leftarrow \boldsymbol{E}}$ transforms a position-vector in terms of the embedded frame $\boldsymbol{E}$ to the corresponding position-vector in terms of the localized frame $\boldsymbol{L}$ of the SRF.

The affine form of the spatial generating function $\boldsymbol{G}_{\text{LE3D}}$ and its inverse provide an affine form for the change of SRF operation between two instances of the Lococentric Euclidean 3D SRF template, SRF_{S} and SRF_{T}, with differing ORMs. This is illustrated in Figure 10.4. In this figure, the Z axis of each of the four frames shown projects out of the page. SRF_{S} has localization parameters $\boldsymbol{q}_{\text{S}}, \boldsymbol{r}_{\text{S}}, \boldsymbol{s}_{\text{S}}$ and associated ORM_{S}, and SRF_{T} has localization parameters $\boldsymbol{q}_{\text{T}}, \boldsymbol{r}_{\text{T}}, \boldsymbol{s}_{\text{T}}$ and associated ORM_{T}. The similarity transformation between these object reference models is denoted by $\boldsymbol{H}_{\text{T} \leftarrow \text{S}}$.

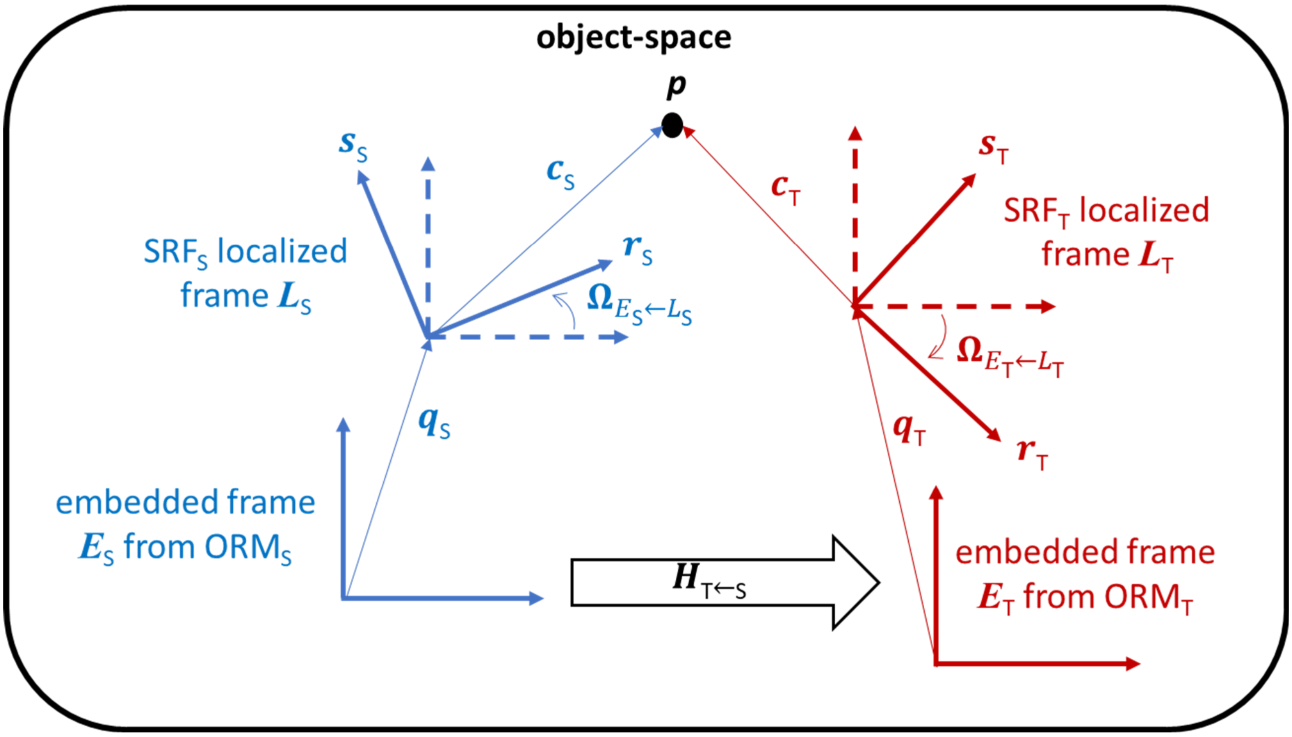

Figure 10.4 — Change of SRF operation for Lococentric Euclidean 3D SRFs

The coordinate $\boldsymbol{c}_{\text{T}}$ in SRF_{T} that corresponds to coordinate $\boldsymbol{c}_{\text{S}}$ in SRF_{S} can be computed using Equation 10.5:

$$\boldsymbol{c}_{\mathrm{T}} = \boldsymbol{G}^{-1}_{\mathrm{LE3D,\,T}} \circ \boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}} \circ \boldsymbol{G}_{\mathrm{LE3D,\,S}}(\boldsymbol{c}_{\mathrm{S}})$$

The change of SRF operation consists of three steps. First, the coordinate $\boldsymbol{c}_{\mathrm{S}}$ is transformed from the localized frame $\boldsymbol{L}_{\mathrm{S}}$ for SRFs to its embedded frame $\boldsymbol{E}_{\mathrm{S}}$, as shown in the left side of Figure 10.4. Next, the similarity transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ transforms the coordinate from the embedded frame $\boldsymbol{E}_{\mathrm{S}}$ for SRFs to the embedded frame $\boldsymbol{E}_{\mathrm{T}}$ for SRFT. Finally, the coordinate is transformed from the embedded frame $\boldsymbol{E}_{\mathrm{T}}$ of SRFT to its localized frame $\boldsymbol{L}_{\mathrm{T}}$, as shown in the right side of Figure 10.4.

Substituting the expression in Equation 10.3 for $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$, and applying the affine transformation to both $\boldsymbol{G}_{\mathrm{LE3D,\,S}}$ and $\boldsymbol{G}^{-1}_{\mathrm{LE3D,\,T}}$ gives:

$$\begin{aligned}
\boldsymbol{c}_{\mathrm{T}} &= \boldsymbol{\Omega}^{\mathrm{T}}_{E_{\mathrm{T}}\leftarrow L_{\mathrm{T}}}\left(\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}\left(\boldsymbol{q}_{\mathrm{S}} + \boldsymbol{\Omega}_{E_{\mathrm{S}}\leftarrow L_{\mathrm{S}}}(\boldsymbol{c}_{\mathrm{S}})\right) - \boldsymbol{q}_{\mathrm{T}}\right)\\
&= \boldsymbol{\Omega}^{\mathrm{T}}_{E_{\mathrm{T}}\leftarrow L_{\mathrm{T}}}\left(\vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{S}} - \boldsymbol{q}_{\mathrm{T}} + \frac{\sigma_{\mathrm{R}\leftarrow\mathrm{S}}}{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{S}}\left(\boldsymbol{q}_{\mathrm{S}} + \boldsymbol{\Omega}_{E_{\mathrm{S}}\leftarrow L_{\mathrm{S}}}(\boldsymbol{c}_{\mathrm{S}})\right)\right)\\
&= \underbrace{\boldsymbol{\Omega}^{\mathrm{T}}_{E_{\mathrm{T}}\leftarrow L_{\mathrm{T}}}\left(\vec{\Delta}_{\mathrm{T}\leftarrow\mathrm{S}} + \frac{\sigma_{\mathrm{R}\leftarrow\mathrm{S}}}{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{S}}\boldsymbol{q}_{\mathrm{S}} - \boldsymbol{q}_{\mathrm{T}}\right)}_{constant\ vector} + \frac{\sigma_{\mathrm{R}\leftarrow\mathrm{S}}}{\sigma_{\mathrm{R}\leftarrow\mathrm{T}}}\underbrace{\left(\boldsymbol{\Omega}^{\mathrm{T}}_{E_{\mathrm{T}}\leftarrow L_{\mathrm{T}}} \circ \boldsymbol{M}_{\mathrm{T}\leftarrow\mathrm{S}} \circ \boldsymbol{\Omega}_{E_{\mathrm{S}}\leftarrow L_{\mathrm{S}}}\right)}_{matrix\ multiplication}(\boldsymbol{c}_{\mathrm{S}})
\end{aligned}$$

where $\boldsymbol{c}_{\mathrm{S}} = (u, v, w)_{\mathrm{S}}$, $\boldsymbol{c}_{\mathrm{T}} = (x, y, z)_{\mathrm{T}}$, and (10.10)

for: i = S or T,

$$\boldsymbol{\Omega}_{E_i\leftarrow L_i} = \begin{bmatrix} r_{i,1} & s_{i,1} & t_{i,1}\\ r_{i,2} & s_{i,2} & t_{i,2}\\ r_{i,3} & s_{i,3} & t_{i,3}\end{bmatrix},$$

$\boldsymbol{r}_i = (r_{i,1} \quad r_{i,2} \quad r_{i,3})$, $\boldsymbol{s}_i = (s_{i,1} \quad s_{i,2} \quad s_{i,3})$, and $\boldsymbol{t}_i = (t_{i,1} \quad t_{i,2} \quad t_{i,3})$ are the CS localization parameters

The advantage of the final form of Equation 10.10 is that it is significantly more efficient when transforming a large number of points, as the constant vector component can be computed only once and reused for each point.

If the corresponding reference transformations of ORMs and ORMT are equivalent, in that they each determine the same embedded frame, Equation 10.7 specializes to Equation 10.11:

$$\begin{aligned}
\boldsymbol{c}_{\mathrm{T}} &= \boldsymbol{G}^{-1}_{\mathrm{LE3D,T}} \circ \boldsymbol{G}_{\mathrm{LE3D,\,S}}(\boldsymbol{c}_{\mathrm{S}}),\\
&= \boldsymbol{\Omega}^{\mathrm{T}}_{E_{\mathrm{T}}\leftarrow L_{\mathrm{T}}}(\boldsymbol{q}_{\mathrm{S}} - \boldsymbol{q}_{\mathrm{T}}) + \boldsymbol{\Omega}^{\mathrm{T}}_{E_{\mathrm{T}}\leftarrow L_{\mathrm{T}}} \circ \boldsymbol{\Omega}_{E_{\mathrm{S}}\leftarrow L_{\mathrm{S}}}(\boldsymbol{c}_{\mathrm{S}}) \qquad (10.11)
\end{aligned}$$

where $\boldsymbol{c}_{\mathrm{S}} = (u, v, w)$

Every Cartesian SRF **C** is equivalent to the LOCOCENTRIC_EUCLIDEAN_3D SRF specified with SRFT localization parameters defined as:

$\boldsymbol{q}$ is the embedded frame vector for the origin of **C**,
$\boldsymbol{r}$ is the unit vector on the primary axis of **C** pointing in the positive direction, and
$\boldsymbol{s}$ is the unit vector on the secondary axis of **C** pointing in the positive direction.

Thus Equations 10.10 and 10.11 apply to Cartesian SRFs as well.

Similarly, the Cartesian coordinate system of any spatial orthonormal frame specified with frame parameter vectors $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ may also be identified with a LOCOCENTRIC_EUCLIDEAN_3D SRF using the same parameters.

10.4.6 Instantiating abstract object-space SRFs

Engineering designs or abstract models are intended for realisation in the physical world or in virtual worlds. Instantiation of such models can require several types of SRFs and specific sequences of position operations.

Abstract models are designed in abstract object-spaces using such Cartesian SRFs as LOCAL_SPACE_RECTANGULAR_3D. In many application domains, abstract models are included in other object-spaces. These other (target) object-spaces may be another abstract object-space, using its own instance of a LOCAL_SPACE_RECTANGULAR_3D SRF, or may be a physical object-space that either uses a Cartesian SRF or a non-Cartesian SRF. To include an abstract model in a target object-space that is specified with a non-Cartesian SRF, it is necessary to establish a localized Cartesian SRF. Whether the target object-space is specified with a Cartesian SRF or a non-Cartesian SRF, the instancing of an abstract model uses a uniform method by establishing a localized Cartesian SRF. The SRF in the target object-space supplies the reference coordinate $\boldsymbol{c}$ to specify the origin of the localized Cartesian SRF, which is instanced from either a LOCAL_TANGENT_SPACE_EUCLIDEAN SRFT or a LOCOCENTRIC_EUCLIDEAN_3D SRFT. In this role, the target object-space SRF becomes the reference SRF and the localized SRF acts as the target SRF.

EXAMPLE 1 A building plan is designed in the source model $\mathrm{SRF_S}$, an abstract space LOCAL_SPACE_RECTANGULAR_3D SRF. A terrestrial site survey establishes the coordinate for the origin of the model in a reference SRF, a celestiodetic $\mathrm{SRF_R}$. The target LOCAL_TANGENT_SPACE_EUCLIDEAN SRF, $\mathrm{SRF_T}$, is instanced at the origin point specified in $\mathrm{SRF_R}$. Source coordinates in $\mathrm{SRF_S}$ are related to local target coordinates in $\mathrm{SRF_T}$ by: $(x_\mathrm{T}, y_\mathrm{T}, z_\mathrm{T}) = \sigma(x_\mathrm{S}, y_\mathrm{S}, z_\mathrm{S})$, where σ is a model scale factor. In addition to scaling, the instanced model is often rotated to adjust its orientation at the instanced position.

NOTE In some modelling applications, the model centre of gravity or bounding box centre, among other choices, is considered to be the "model origin". However, for purposes of model instantiation, the model origin is the point with coordinate (0, 0, 0) in the SRF in which the model is defined.

The instancing of an abstract model entails the following steps, which provide a uniform method for both Cartesian and non-Cartesian-based SRFs of target object-space:

1) If the abstract space geometric model is specified in $\mathrm{SRF_M}$ that is not a LOCAL_SPACE_RECTANGULAR_3D SRF and is instead specified in another 2D or 3D abstract space SRF, the model is converted using Equation 10.6 from $\mathrm{SRF_M}$ to $\mathrm{SRF_S}$, a LOCAL_SPACE_RECTANGULAR_3D SRF. Otherwise, $\mathrm{SRF_M}$ becomes $\mathrm{SRF_S}$.

2) The position at which the model is instanced in the physical or abstract target object-space is identified by a coordinate $\boldsymbol{c}$ in $\mathrm{SRF_R}$, a reference SRF for the target object-space.

3) The target for the conversion is $\mathrm{SRF_T}$, a localized Cartesian 3D SRF with its origin specified by the coordinate $\boldsymbol{c}$ in $\mathrm{SRF_R}$. $\mathrm{SRF_T}$ must be compatible with $\mathrm{SRF_R}$. $\mathrm{SRF_T}$ may be either a LOCAL_TANGENT_SPACE_EUCLIDEAN SRF or a LOCOCENTRIC_EUCLIDEAN_3D SRF. $\mathrm{SRF_T}$ is realised by the reference coordinate $\boldsymbol{c}$ and the $\mathrm{SRF_T}$ template parameters.

4) A world transformation is supplied to correctly position, scale, and orient the geometric model instance. The transformation includes a scaled rotation matrix $\sigma\boldsymbol{R}$, where σ is a scale factor and $\boldsymbol{R}$ is a rotation or identity matrix. The transformation may also optionally include $\vec{\Delta}_\mathrm{T} = (\Delta x_\mathrm{T}, \Delta y_\mathrm{T}, \Delta z_\mathrm{T})$, an offset of the model origin from the $\mathrm{SRF_T}$ origin.

5) Each model vertex coordinate in $\mathrm{SRF_S}$ is converted to a corresponding coordinate in $\mathrm{SRF_T}$ through the following transformation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_T = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}_\mathrm{T} + \sigma\boldsymbol{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_S . \tag{10.12}$$

This equation is in the form of Equation 10.4 where $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}(\boldsymbol{v}) = \vec{\Delta}_\mathrm{T} + \sigma\boldsymbol{R}\boldsymbol{v}$ and $\boldsymbol{G}_\mathrm{S} = \boldsymbol{G}_\mathrm{T} =$ Identity, thus the conversion may be viewed as a change of SRF operation. See also 10.5.3 Example. In the terminology of ISO/IEC 18023-1 Data Representation Model (DRM) classes, $\boldsymbol{W} = \sigma\boldsymbol{R}$ is the world transformation 3x3 matrix class.

NOTE Equation 10.12 illustrates that digital graphic composite pattern modelling techniques such as SceneGraph trees that use scale and rotation matrices $\boldsymbol{W}$ together with translation operations at each tree node are special cases of Equation 10.4. See also 10.5.3 Example.

EXAMPLE 2 A model geometry is specified in $\mathrm{SRF_M}$, an abstract object-space Cartesian SRF. The model is to be instanced in a physical object-space with a geocentric reference SRF, $\mathrm{SRF_R}$. The $\mathrm{SRF_R}$ reference coordinate $\boldsymbol{c}$ determines the position of the model origin at point $\boldsymbol{q}$. $\mathrm{SRF_T}$ is the target SRF realised from the LOCOCENTRIC_EUCLIDEAN_3D

SRFT with template parameters $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, where $\boldsymbol{r}$ and $\boldsymbol{s}$ are, respectively, the primary and secondary coordinate axis unit vectors of SRF_T with respect to SRF_R. The orientation of the instanced model with respect to SRF_T (and SRF_R) is determined by the model's rotation matrix $\boldsymbol{R}$.

EXAMPLE 3 A CAD model of an automobile wheel is designed in an abstract object-space using a LOCAL_SPACE_RECTANGULAR_3D source SRF, SRF_S. The wheel model is then instanced into another abstract object-space where a CAD model for the entire car is being designed, using a target LOCAL_SPACE_RECTANGULAR_3D SRF, SRF_T. This is illustrated in Figure 10.5. In this simple case, there is no need for a distinct reference SRF, and there is no need to localize the target SRF. The centre of the wheel model is at the origin of SRF_S, and its orientation is aligned with the axes of SRF_S. A transformation $\boldsymbol{H}_{T \leftarrow S}$ embeds an instance of the SRF_S into the abstract object-space of the car model, scaling the wheel model by σ to be consistent with the car model, and translating it by $\boldsymbol{\Delta}_T$ to the appropriate position with respect to the car model. If necessary, the transformation can also rotate the wheel model by $\boldsymbol{R}$ to align it with the car model.

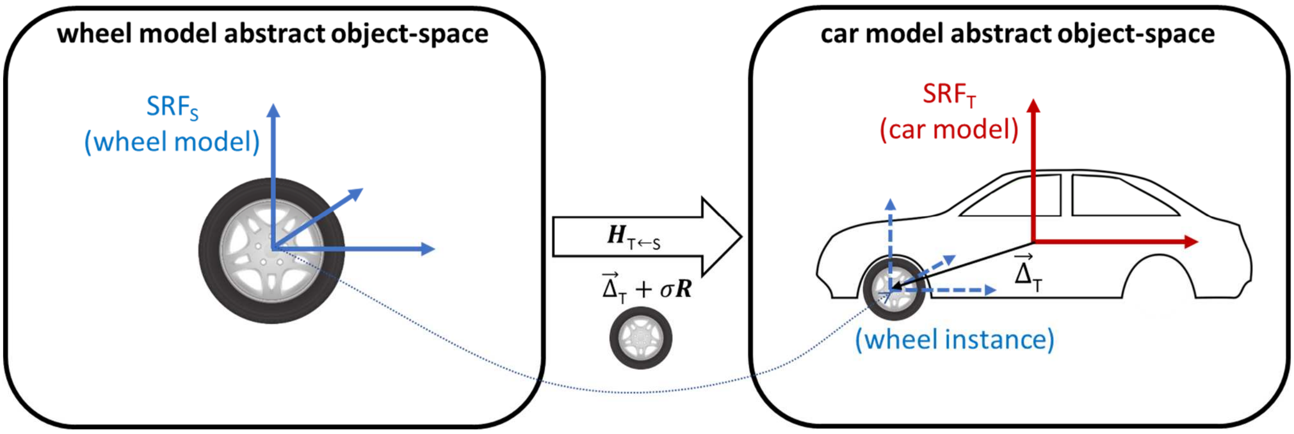

Figure 10.5 — Abstract object realised in another abstract object-space

EXAMPLE 4 A house is designed in an abstract object-space using a LOCAL_SPACE_RECTANGULAR_3D source SRF, SRF_S. A terrestrial site survey using the GEODETIC_WGS_1984 SRF as the reference SRF, SRF_R, establishes the geodetic coordinate $(\lambda_0, \varphi_0, h_0)$ of the southeast corner of the site where the house will be built. This geodetic coordinate provides parameter values for an instance of the LOCAL_TANGENT_SPACE_EUCLIDEAN SRF template that defines the origin (and tangent point) of the target SRF, SRF_T. The origin of SRF_T is at the southeast corner of the building site, and its axes align with local east and local north. Because the house will not be positioned at the origin of SRF_T, or aligned with its axes, the transformation $\boldsymbol{H}_{T \leftarrow S}$ scales the house model to its actual size, rotates it to its planned orientation on the site, and translates it to its planned position. This is illustrated in Figure 10.6.

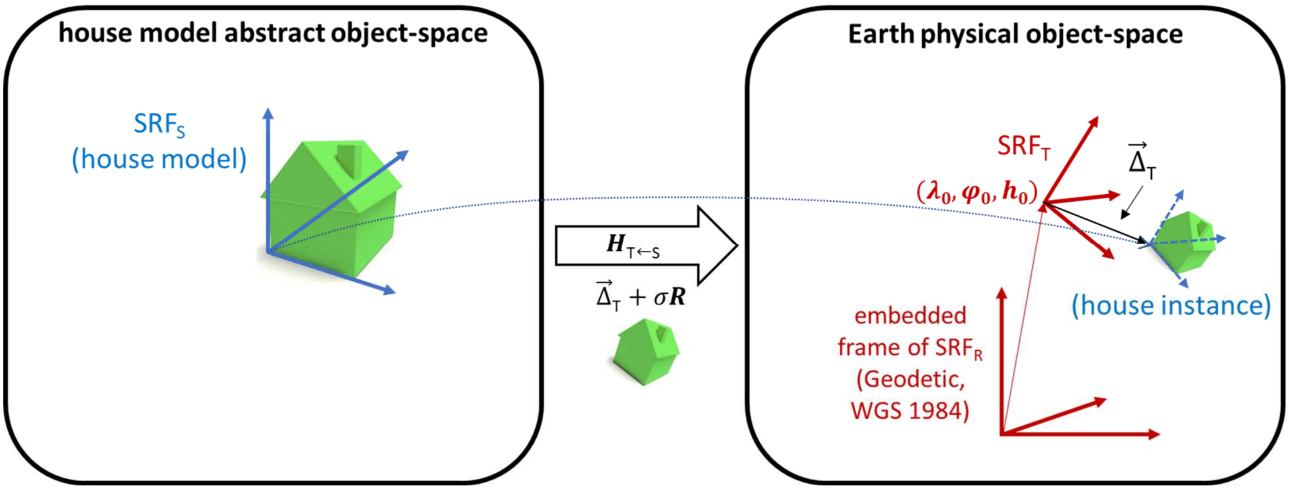

Figure 10.6 — Abstract object realised using a geodetic reference point

EXAMPLE 5 A house is designed in an abstract object-space using a LOCAL_SPACE_RECTANGULAR_3D source SRF, SRF_S. The GEOCENTRIC_WGS_1984 SRF is used as the reference SRF, SRF_R. The localization parameters $\boldsymbol{q}_T$, $\boldsymbol{r}_T$, and $\boldsymbol{s}_T$, for an instance of the LOCOCENTRIC_EUCLIDEAN_3D SRF template define the target SRF, SRF_T,. The geocentric coordinate (x, y, z) determines a corner of the building site, which is the origin of SRF_T at $\boldsymbol{q}_T$. The building site footprint determines the axes $\boldsymbol{r}_T$ and $\boldsymbol{s}_T$. Because the house will not be positioned at the origin of SRF_T, or aligned with its axes, the transformation $\boldsymbol{H}_{\mathrm{T}\leftarrow\mathrm{S}}$ scales the house model to its actual size, rotates it to its planned orientation on the site, and translates it to its planned position. This is illustrated in Figure 10.7.

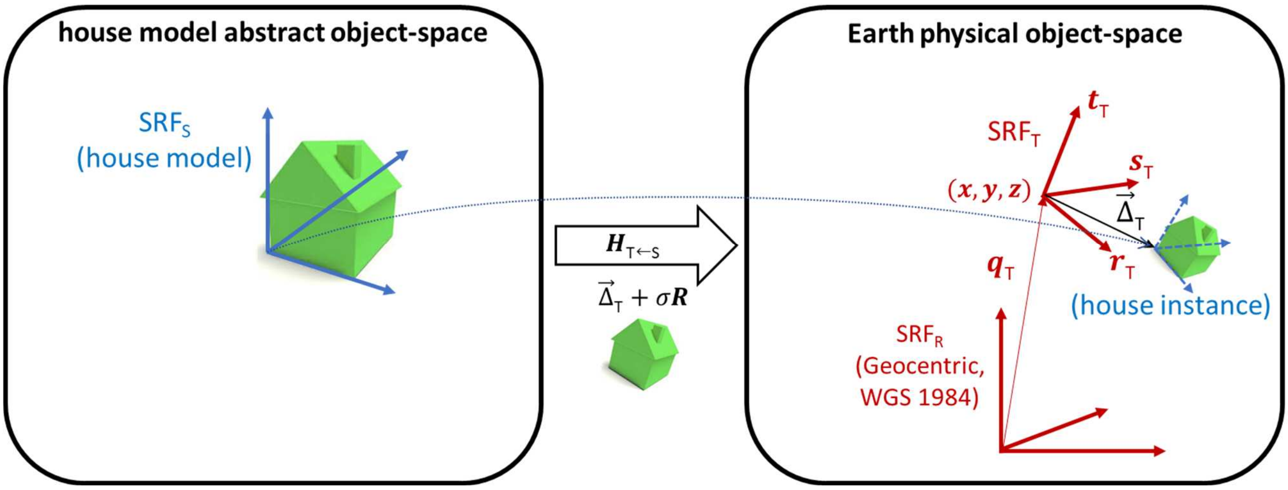

Figure 10.7 — Abstract object realised using a geocentric reference point

10.5 Vector operations

10.5.1 Introduction

Directions and vector quantities associated with a 3D SRF are specified with respect to 3D orthonormal frames (see 5.2.3). Given an orthonormal frame for object-space (see 8.4.4), a direction is represented as a unit vector in the Cartesian coordinate system determined by the frame. Vector quantities, such as velocity or force, are specified as vectors of appropriate direction and magnitude in the frame. The 3D orthonormal frame is termed the vector reference frame (see 5.3.6.4).

The choice of the vector reference frame is often determined by the requirements of the user application. A 3D SRF can be used to directly or indirectly specify the vector reference frame.

One choice is to use a local tangent frame as the vector reference frame. A coordinate $\boldsymbol{c}$ in the interior of the domain of an orthogonal right-handed 3D[28] SRF specifies the local tangent frame at $\boldsymbol{c}$ (see 5.3.6.3 and 8.4.5). This local tangent frame has its origin at $\boldsymbol{c}$ and its basis vectors tangent to the coordinate-component curves of the 3D SRF at the origin.

In the special case that the 3D SRF is a Cartesian SRF, all coordinate-component curves are straight lines in object-space. For each coordinate-component, all coordinate-component curve instances are parallel to one another. All coordinate-component curve instances for each coordinate-component are perpendicular to the coordinate-component curve instances for the other two coordinate-components. Thus, all local tangent frames of a Cartesian SRF are oriented the same way and differ only in the location of the frame origins.

A second, less restrictive, choice is to use a localized frame (see 8.4.5) as the vector reference frame. A Cartesian SRF with spatial generating function $\boldsymbol{G}(\,)$ specifies a localized frame by coordinates $\boldsymbol{c}_o, \boldsymbol{c}_r, \boldsymbol{c}_s$ where $\boldsymbol{G}(\boldsymbol{c}_o)$ is the localized frame origin and the vectors $\boldsymbol{G}(\boldsymbol{c}_r) - \boldsymbol{G}(\boldsymbol{c}_o)$ and $\boldsymbol{G}(\boldsymbol{c}_s) - \boldsymbol{G}(\boldsymbol{c}_o)$, which are perpendicular to each other, are its basis vectors (see 5.3.6.3).

Given a vector with respect to one vector reference frame, the representation of the vector can be converted to a second vector reference frame if the orientation of one frame with respect to the other can be computed. The conversion computation in various situations is treated in 10.5.2 and 10.5.3. This operation uses a specialized

[28] All of the 3D SRFTs in this International Standard are based on orthogonal right-handed CSs.

form of Equation 10.3, dropping the translation term since vectors are translation invariant and dropping the scale factor to preserve the magnitude of the vector.

10.5.2 Representing vectors in different vector reference frames

Given a source SRF, SRF_S, with corresponding ORM_S and embedded frame E_S, and a target SRF, SRF_T, in the same object-space, with a corresponding ORM_T and embedded frame E_T, a vector reference frame F_S can be derived from SRF_S at coordinate $\boldsymbol{c}_S$ as described in 10.5.1. Similarly, another vector reference frame F_T can be derived from SRF_T at coordinate $\boldsymbol{c}_T$.

There are several conditions under which the embedded frames E_S and E_T are the same (i.e., share the same origin and the same basis vectors):

a) SRF_S and SRF_T are the same SRF,
b) SRF_S and SRF_T are specified using the same ORM, or
c) SRF_S and SRF_T are specified using different ORMs that determine normal embeddings that produce the same embedded frame.

Given $\boldsymbol{v}_S$, a vector with magnitude and direction represented in vector reference frame F_S, the same vector, denoted as $\boldsymbol{v}_T$, can be represented with respect to vector reference frame F_T as:

$$\boldsymbol{v}_T = \boldsymbol{\Omega}_{T\leftarrow S}\ \boldsymbol{v}_S,$$

where $\boldsymbol{\Omega}_{T\leftarrow S}$ is the orientation of vector reference frame F_S with respect to vector reference frame F_T (see 6.3.2).

When both vector reference frames F_S and F_T are specified using the same embedded frame, $\boldsymbol{\Omega}_{T\leftarrow S}$ is the direction cosine matrix that transforms positions in F_S to equivalent positions in F_T (see 6.2.2):

$$\boldsymbol{\Omega}_{T\leftarrow S} = \begin{bmatrix} \boldsymbol{r}_S \bullet \boldsymbol{r}_T & \boldsymbol{s}_S \bullet \boldsymbol{r}_T & \boldsymbol{t}_S \bullet \boldsymbol{r}_T \\ \boldsymbol{r}_S \bullet \boldsymbol{s}_T & \boldsymbol{s}_S \bullet \boldsymbol{s}_T & \boldsymbol{t}_S \bullet \boldsymbol{s}_T \\ \boldsymbol{r}_S \bullet \boldsymbol{t}_T & \boldsymbol{s}_S \bullet \boldsymbol{t}_T & \boldsymbol{t}_S \bullet \boldsymbol{t}_T \end{bmatrix}, \tag{10.13}$$

where: $\boldsymbol{r}_i, \boldsymbol{s}_i$ and $\boldsymbol{t}_i$ are the basis vectors of vector reference frame F_i at $\boldsymbol{c}_i$, for i = S, T.

When the two vector reference frames F_S and F_T are specified using different embedded frames, $\boldsymbol{v}_T$ is computed as:

$$\boldsymbol{v}_T = \boldsymbol{R}_T^{T} \circ \boldsymbol{M}_{T\leftarrow S} \circ \boldsymbol{R}_S\ \boldsymbol{v}_S \tag{10.14}$$

where:

$\boldsymbol{R}_T^{T}$ is the transpose of $\boldsymbol{R}_T$,

$\boldsymbol{M}_{T\leftarrow S}$ is the rotation matrix component of the similarity transformation $\boldsymbol{H}_{T\leftarrow S}$ from ORM_S to ORM_T (see Equation 10.3 in 10.3.2) and

for: i = S or T,

$$\boldsymbol{R}_i = \begin{bmatrix} r_{i,1} & s_{i,1} & t_{i,1} \\ r_{i,2} & s_{i,2} & t_{i,2} \\ r_{i,3} & s_{i,3} & t_{i,3} \end{bmatrix},$$

$\boldsymbol{r}_i = (r_{i,1} \quad r_{i,2} \quad r_{i,3})$, $\boldsymbol{s}_i = (s_{i,1} \quad s_{i,2} \quad s_{i,3})$, and $\boldsymbol{t}_i = (t_{i,1} \quad t_{i,2} \quad t_{i,3})$ are the basis vectors for the vector reference frame at $\boldsymbol{c}_i$ with respect to the embedded frame E_i.

Equation 10.14 is derived from Equation 10.3 by dropping the translation term since vectors are translation invariant and dropping the scale factor ${}^{\sigma_{SR}}/_{\sigma_{TR}}$ to preserve the magnitude of the vector.

The rotation matrix $\boldsymbol{M}_{T\leftarrow S}$ in Equation 10.14 is termed the *orientation of SRF_S at reference coordinate $\boldsymbol{c}_S$, with respect to SRF_T at reference coordinate $\boldsymbol{c}_T$*. The rotation matrix $\boldsymbol{M}_{T\leftarrow S}$ is a generalization of the matrix in Equation 10.13 that accounts for the change of embedded frames between ORM_S and ORM_T.

EXAMPLE SRF_S is SRF GEODETIC_WGS_1984 and SRF_T is SRF GEOCENTRIC_WGS_1984. With SRF_S reference coordinate $\boldsymbol{c}_S = (\lambda, \ \varphi, \ h) = (-77\pi/180, \ +38{,}88\pi/180, \ 0)$. The Washington monument, an obelisk located at $\boldsymbol{c}_S$, points approximately in the direction $\boldsymbol{n}_{\mathrm{S}} = (0,\quad 0,\quad 1)$ in the local tangent frame at $\boldsymbol{c}_{\mathrm{S}}$. In this example, $\mathrm{ORM}_{\mathrm{S}} = \mathrm{ORM}_{\mathrm{T}}$ so that case b) applies. Since $\mathrm{SRF}_{\mathrm{T}}$ is a Cartesian SRF, the local tangent frame at a coordinate $\boldsymbol{c}_{\mathrm{T}}$ in $\mathrm{SRF}_{\mathrm{T}}$ has the same basis vectors as the embedded frame, hence the dot product components of $\boldsymbol{R}$ appearing in Equation 10.13 with the basis vectors $\boldsymbol{r}_{\mathrm{S}}, \boldsymbol{s}_{\mathrm{S}}, \boldsymbol{t}_{\mathrm{S}}$ for the tangent frame at $\boldsymbol{c}_{\mathrm{S}}$ reduce to column vectors for $\boldsymbol{r}_{\mathrm{S}}, \boldsymbol{s}_{\mathrm{S}}, \boldsymbol{t}_{\mathrm{S}}$ in embedded frame coordinates, so that:

$$\boldsymbol{n}_{\mathrm{T}} = \boldsymbol{R}_{\mathrm{S}}\, \boldsymbol{n}_{\mathrm{S}} = \begin{bmatrix} r_{\mathrm{S},1} & s_{\mathrm{S},1} & t_{\mathrm{S},1} \\ r_{\mathrm{S},2} & s_{\mathrm{S},2} & t_{\mathrm{S},2} \\ r_{\mathrm{S},3} & s_{\mathrm{S},3} & t_{\mathrm{S},3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_{\mathrm{S},1} \\ t_{\mathrm{S},2} \\ t_{\mathrm{S},3} \end{bmatrix} = \boldsymbol{t}_{\mathrm{S}}.$$

Then using the expression in 8.4.4 Example 3 for $\boldsymbol{t}_{\mathrm{S}}$:

$$\begin{aligned} \boldsymbol{t}_{\mathrm{S}} &= (\cos\lambda_0 \cos\varphi_0 \quad \sin\lambda_0 \cos\varphi_0 \quad \sin\varphi_0) \\ &= (\cos(-77\pi/180)\cos(38{,}88\pi/180) \quad \sin(-77\pi/180)\cos(38{,}88\pi/180) \quad \sin(38{,}88\pi/180)) \\ &= (0{,}17511592 \quad -0{,}75851036 \quad 0{,}62769136). \end{aligned}$$

The resulting vector $\boldsymbol{n}_{\mathrm{T}} = (0{,}17511592 \quad -0{,}75851036 \quad 0{,}62769136)$ is the direction vector at coordinate $\boldsymbol{c}_{\mathrm{T}}$ in $\mathrm{SRF}_{\mathrm{T}}$.

10.5.3 Instantiating abstract object-space SRF directions in another object-space

Engineering designs and abstract models are often intended for realization in the physical world. In such cases, the operation of changing the representation of direction vector $\boldsymbol{n}_{\mathrm{S}}$ in a linear SRF representing the abstract space to a direction vector $\boldsymbol{n}_{\mathrm{T}}$ with respect to a vector reference frame F_{T} in an SRF for a physical object-space is based on Equation 10.11. Denoting the basis vectors for F_{T} by $\boldsymbol{r}_{\mathrm{T}}, \boldsymbol{s}_{\mathrm{T}}, \boldsymbol{t}_{\mathrm{T}}$, and transforming the direction vector through matrix multiplication by a given invertible matrix 3x3 $\boldsymbol{W}$ (see 10.4.6 Example), $\boldsymbol{n}_{\mathrm{T}}$ is computed as:

$$\begin{aligned} \widetilde{\boldsymbol{n}}_{\mathrm{S}} &= \frac{1}{|\mathbf{W}|}\boldsymbol{W}\boldsymbol{n}_{\mathrm{S}} \\ \boldsymbol{n}_{\mathrm{T}} &= \boldsymbol{R}_{\mathrm{T}}\widetilde{\boldsymbol{n}}_{\mathrm{S}} \end{aligned} \qquad (10.15)$$

where matrix $\boldsymbol{R}_{\mathrm{T}}$ is defined in Equation 10.14. Division by the determinant $|\boldsymbol{W}|$ cancels any scaling by matrix $\boldsymbol{W}$ to ensure that $\boldsymbol{n}_{\mathrm{T}}$ is a unit vector. (The rotation matrix $\boldsymbol{R}_{\mathrm{T}}$ does not change the length of $\widetilde{\boldsymbol{n}}_{\mathrm{S}}$.)

EXAMPLE In ISO/IEC 18023-1, if an instance of the class <DRM Geometry Model Instance> has a component of class <DRM World Transformation>, that component specifies an invertible matrix $\boldsymbol{W}$ and a coordinate $\boldsymbol{c}$ in the <DRM Environment Root> SRF. If $\boldsymbol{n}_{\mathrm{S}}$ is a direction vector at reference coordinate $\boldsymbol{c}_{\mathrm{S}}$ in an associated LOCAL_SPACE_RECTANGULAR_3D <DRM Geometry Model>, Equation 10.11 may be used to compute $\boldsymbol{c}_{\mathrm{T}}$ in the <DRM Environment Root> SRF and Equation 10.15 may be used to compute the $\boldsymbol{n}_{\mathrm{T}}$ direction at $\boldsymbol{c}_{\mathrm{T}}$. This procedure to change <DRM Geometry Model> coordinates and directions to the environment root SRF is termed "model instancing".

10.6 Euclidean distance operation

This International Standard supports an operation to return the Euclidean distance between two object-space locations using the coordinates of those locations in an SRF.

If $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are two coordinates in an SRF, and if $\boldsymbol{G}$ is the generating function of the CS of the SRF, the *Euclidean distance* d_{E} between the corresponding points in object-space is given by:

$$d_E(\boldsymbol{c}_1, \boldsymbol{c}_2) = d\big(\boldsymbol{G}(\boldsymbol{c}_1), \boldsymbol{G}(\boldsymbol{c}_2)\big)$$

where d is the Euclidean metric.

10.7 Geodesic distance operations

10.7.1 Introduction

A curve on a smooth surface that has the property that any sufficiently small segment of it realizes the shortest distance on the surface between the segment’s two endpoints is termed a geodesic (see Figure 10.8). The formal definition of a geodesic is given in A.7.4.

EXAMPLE 1 On a sphere, the equator, the meridians, and all other great circles are geodesics. Likewise any segment of one of these curves is a geodesic. No parallel of latitude except the equator is a geodesic.

EXAMPLE 2 On an oblate ellipsoid, the equator is a geodesic, and the meridians are all geodesics. All the other geodesics are curves which wind around the ellipsoid between two parallels of opposite latitude and any segment of which that crosses the equator, crosses at some non-right angle.

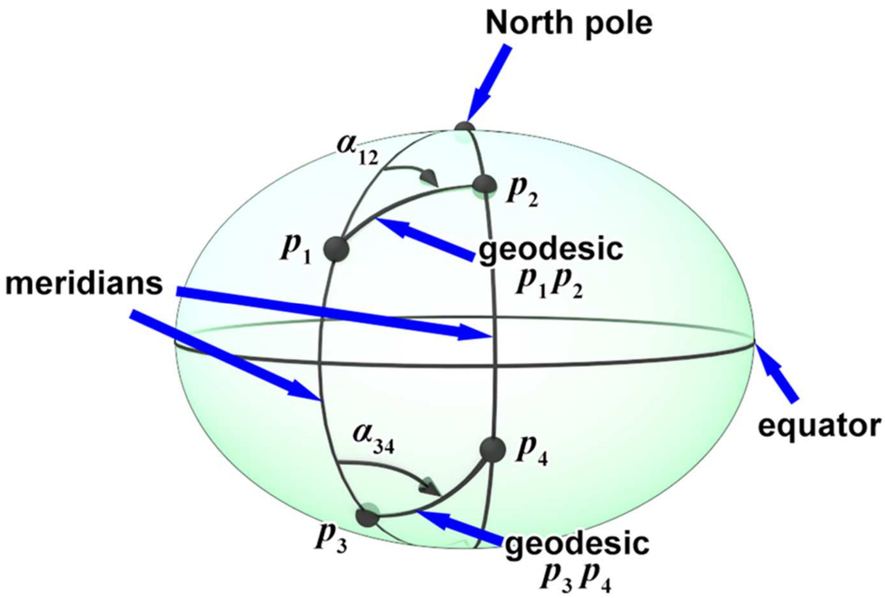

Figure 10.8 — Examples of geodesics

Let points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ lie on a smooth surface. The shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ is the shortest arc length associated with any of the smooth surface curves that connect $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. This distance is unique, but the curve that has this arc length may not be unique. In particular, for the two pole points, every meridian is such a shortest curve.

EXAMPLE 3 On an oblate ellipsoid, let $\boldsymbol{p}_1$ be the point with surface geodetic coordinates (λ, φ) = (0°, 20°) and let $\boldsymbol{p}_2$ be the point diametrically opposite, *i.e.,* with surface geodetic coordinates (λ, φ) = (180°, -20°). In that case, the shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ is twice the meridional quadrant, *i.e.*, twice the length of a meridian from equator to pole. But there are two distinct curves from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ which have this number as their arc length – one passes through the north pole and the other passes through the south pole. (Both are composed of segments of meridians).

EXAMPLE 4 On an oblate ellipsoid with eccentricity ε, let points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ lie on the equator but be separated by a longitude difference that is less than π and more than $\pi\sqrt{1-\varepsilon^2}$, an angle termed the "lift-off longitude". Then there will be two curves from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ whose arc length is the shortest distance from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ – one lying in the northern hemisphere, the other lying (symmetrically) in the southern hemisphere. If the longitude difference is less than $\pi\sqrt{1-\varepsilon^2}$, the shorter equator segment from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ is the shortest connecting curve.

If a curve lying on a smooth surface connects point $\boldsymbol{p}_1$ to point $\boldsymbol{p}_2$, and if that curve's arc length is also the shortest distance from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$, then that curve is a geodesic. Thus, the arc length of the shortest curve connecting the two points is termed the *geodesic distance*.

EXAMPLE 5 The two curves from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ defined in Example 3 are geodesics.

EXAMPLE 6 The two curves from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ defined in Example 4 are geodesics.

The converse is not true. If a geodesic starts at point $\boldsymbol{p}_1$ and ends at point $\boldsymbol{p}_2$, its arc length may or may not be the same as the shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$.

EXAMPLE 7 Let points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ lie on the equator of a sphere or oblate ellipsoid at longitudes 0° and 181°, respectively. The segment of the equator from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ that is continuous in longitude from 0° to 181° is a geodesic. (All segments of the equator are geodesics). However, its arc length is not the shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. Any curve which realizes the shortest distance on the surface from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ has to lie within a single hemisphere of longitude.

There are two problems of interest pertaining to geodesics on an oblate ellipsoid. In the first, termed the direct problem, a surface point, an azimuth, and a distance are given. The problem is to find a second surface point which terminates the (unique) geodesic whose initial point is the given point, whose initial forward azimuth is the given azimuth, and whose arc length is the given distance. Also to be found is the geodesic's terminal forward azimuth. The details are given in 10.7.3.

In the second problem, termed the indirect problem, two distinct surface points are given. The problem is to find the shortest distance on the surface between the two given points, and find the set of curves (which will be geodesics) whose arc lengths equal this shortest distance. In addition, the initial and terminal forward azimuths of each curve is to be found. The details are in 10.7.4.

This International Standard supports the geodesic operations for SRFs based on SRFT CELESTIODETIC, PLANETODETIC, and all map projection SRFTs.

Given two surface coordinates $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, respectively, the *geodesic distance operation*:

$$s = d_G(\boldsymbol{c}_1, \boldsymbol{c}_2)$$

is defined as the distance solution to the indirect problem for (λ_1, φ_1) and (λ_2, φ_2) where (λ_1, φ_1) is the surface geodetic coordinate for $\boldsymbol{c}_1$ and (λ_2, φ_2) is the surface geodetic coordinate for $\boldsymbol{c}_2$.

An extended version of this operation provides the forward azimuth value α_1 at $\boldsymbol{c}_1$ and the forward azimuth value α_2 at $\boldsymbol{c}_2$:

$$\{s, \alpha_1, \alpha_2\} = d_{GI}(\boldsymbol{c}_1, \boldsymbol{c}_2).$$

The geodesic destination operation corresponding to the direct problem requires a starting point $\boldsymbol{c}_1$, a forward azimuth value α_1, at $\boldsymbol{c}_1$ and a positive distance s. It returns the destination point $\boldsymbol{c}_2$ and the forward azimuth value α_2 at $\boldsymbol{c}_2$:

$$\{\boldsymbol{c}_2, \alpha_2\} = d_{GD}(\boldsymbol{c}_1, \alpha_1, s)$$

where $\{(\lambda_2, \varphi_2), \alpha_2\}$ is the direct problem solution for input parameter values $\{(\lambda_1, \varphi_1), \alpha_1, s\}$.

There is a large body of literature concerning computational techniques to solve the direct and indirect problems. In the interest of accuracy and computational efficiency, many of these computational techniques treat the problems by sub-cases -- short lines, long lines, intermediate length lines, and other caveats and exceptions. Each of these has been optimized in a way that is appropriate for the intended application or user domain. For purposes of this International Standard, a recently published treatment ([ROL10]) that has one mathematical formulation to cover all cases is utilized.

10.7.2 Auxiliary functions

The treatment of the direct and indirect problems in 10.7.3 and 10.7.4 require the auxiliary functions defined in this subclause.

An important characteristic of a geodesic on an oblate ellipsoid is that the quantity termed the (non-metric) *Clairaut constant* and defined by:

$$c = \frac{\sin(\alpha)\cos(\varphi)}{\sqrt{1 - \varepsilon^2 \sin^2(\varphi)}}$$

has a constant value at every point on a given geodesic, where (λ, φ) is the coordinate of a point on the geodesic and α is the azimuth of the curve at that point.

The mathematics required to solve the direct and indirect problems involves the use of elliptic integrals. The incomplete elliptic integral of third kind is defined for real n, θ and m, with $m^2 < 1$ as:

$$P(n,\theta,m) = \int_0^\theta d\xi \Big/ \left((1 - n\,sin^2\,\xi)\sqrt{1 - m\,sin^2\,\xi}\right).$$

The treatment in [ROL10] defines two auxiliary functions: a longitude difference function $\mathrm{L}(c,\theta_1,\theta_2)$ and an arc length function $\mathrm{A}(c,\theta_1,\theta_2)$ that are defined for all values of c, θ_1 and θ_2 by:

$$\begin{aligned}\mathrm{L}(c,\theta_1,\theta_2) &= \left(c(1-\varepsilon^2)/\sqrt{1-c^2\varepsilon^2}\right)\left(P(k^2(c),\theta_2,k^2(c)\varepsilon^2)\right.\\ &\qquad \left.- P(k^2(c),\theta_1,k^2(c)\varepsilon^2)\right),\ c \neq 0\\ \mathrm{L}(0,\theta_1,\theta_2) &= \lim_{c\to 0^+} \mathrm{L}(c,\theta_1,\theta_2).\end{aligned} \tag{10.16}$$

and

$$\begin{aligned}\mathrm{A}(c,\theta_1,\theta_2) &= \left(a(1-\varepsilon^2)/\sqrt{1-c^2\varepsilon^2}\right)\left(P(k^2(c)\varepsilon^2,\theta_2,k^2(c)\varepsilon^2)\right.\\ &\qquad \left.- P(k^2(c)\varepsilon^2,\theta_1,k^2(c)\varepsilon^2)\right).\end{aligned} \tag{10.17}$$

where:

$$k^2(c) = (1-c^2)/(1-c^2\varepsilon^2).$$

10.7.3 The direct problem

Given an oblate ellipsoid with major semi-axis a and eccentricity ε , let $\boldsymbol{p}_1$ be a non-polar point on the ellipsoid given by its surface geodetic coordinates (λ_1, φ_1). Let a geodesic be defined with $\boldsymbol{p}_1$ as its initial point, α_1 as its initial forward azimuth, and arc length s. This geodesic will terminate at a point $\boldsymbol{p}_2$.

The direct problem requires finding the surface geodetic coordinates (λ_2, φ_2) of $\boldsymbol{p}_2$ and the forward azimuth α_2 of the geodesic at the point $\boldsymbol{p}_2$. The quantity $\alpha_2 + \pi$ is termed the *back azimuth* at $\boldsymbol{p}_2$ as it points backwards toward $\boldsymbol{p}_1$.

The given parameters are restricted to $-\pi/2 < \varphi_1 < \pi/2,\ \ -\pi < \alpha_1 \leq \pi,$ and $s > 0$.

The functions $\mathrm{L}(c,\theta_1,\theta_2)$and $\mathrm{A}(c,\theta_1,\theta_2)$ are used to solve the direct problem.

The given values in the direct problem (λ_1, φ_1) and α_1 determine the Clairaut constant c,

$$c = \sin(\alpha_1)\cos(\varphi_1)/\sqrt{1-\varepsilon^2\,sin^2(\varphi_1)}.$$

Then using the longitude difference function,

$$\lambda_2 = \lambda_1 + \mathrm{L}(c,\theta_1,\theta_2),$$

$$\varphi_2 = \arcsin(k(c)\sin\theta_2)\text{ , and}$$

$$\alpha_2 = \operatorname{arctan2}\left(c\sqrt{1-k^2(c)\sin^2\theta_2}\,, k(c)\sqrt{1-c^2\varepsilon^2}\cos\theta_2\right)$$

where:

$$\theta_1 = \arcsin(\sin(\varphi_1)/k(c))\text{ ,}$$

$$k(c) = \pm\sqrt{\frac{1-c^2}{1-c^2\,\varepsilon^2}},\ \ k(c) \geq 0 \text{ if } |\alpha_1| \leq \pi/2 \text{ and } k < 0 \text{ otherwise,}$$

and θ_2 is determined by the arc length function:

$$s = \mathrm{A}(c,\theta_1,\theta_2). \tag{10.18}$$

Equation 10.18 has a unique solution for θ_2, which can be found by iterative methods.

10.7.4 The indirect problem

Given an oblate ellipsoid with major semi-axis a and eccentricity ε, let $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ be two points on the ellipsoid given by their surface geodetic coordinates (λ_1, φ_1) and (λ_2, φ_2).

The indirect problem requires finding the shortest distance s on the ellipsoid from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. Further, for each curve from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ whose arc length is s, it is required to find the forward azimuths α_1 and α_2 at the points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ respectively. (Such curves will be geodesics, and there will be 1, 2, or infinitely many of them).

The given parameters are restricted to $-\pi \le \lambda_2 - \lambda_1 \le \pi,\ -\pi/2 \le \varphi_1 \le \pi/2$, and $-\pi/2 \le \varphi_2 \le \pi/2$.

The solution to the indirect problem can be determined once c, the Clairaut constant for the solution geodesic curve segment, is found. Dealing with the extreme c values 0 and 1 separately simplifies the process.

The single meridional case: $c = 0$ if $\lambda_2 = \lambda_1$ or if either point is a pole ($|\varphi_1| = \pi/2$ or $|\varphi_2| = \pi/2$). Then if $\varphi_1 < \varphi_2$, the solution is:

$$s = \mathrm{A}(0, \varphi_1, \varphi_2)\text{, and } \alpha_1 = \alpha_2 = 0.$$

Otherwise $\phi_1 > \phi_2$, and the solution is:

$$s = \mathrm{A}(0, \varphi_2, \varphi_1)\text{, and } \alpha_1 = \alpha_2 = \pi.$$

If either point is a pole, the azimuth at that point is undefined. The solution geodesic curve segment is unique unless both given points are poles. In that case the solution set is the infinite set of all meridians.

Meridional segments joined at pole: $c = 0$ if $\lambda_2 = \lambda_1 \pm \pi$ and $\varphi_2 \ge -\varphi_1$. Then

$$s = \mathrm{A}(0, \varphi_1, \pi - \varphi_2), \alpha_1 = 0,\ \alpha_2 = \pi$$

and the geodesic curve segment passes through the north pole.

Similarly, $c = 0$ if $\lambda_2 = \lambda_1 \pm \pi$ and $\varphi_2 < -\varphi_1$. Then

$$s = -\mathrm{A}(0, \varphi_1, -\varphi_2 - \pi), \alpha_1 = \pi,\ \alpha_2 = 0$$

and the geodesic curve segment passes through the south pole.

Eastward Equatorial segment: $c = 1$ if $\varphi_1 = \varphi_2 = 0$ and $0 < \lambda_2 - \lambda_1 \le \pi\sqrt{1-\varepsilon^2}$. Then

$$s = a(\lambda_2 - \lambda_1)\text{, and } \alpha_1 = \alpha_2 = \pi/2$$

The solution is unique.

Nearly antipodal Eastward Equatorial segment:

If $\varphi_1 = \varphi_2 = 0$ and the points are separated by more than the lift-off longitude ($\pi\sqrt{1-\varepsilon^2} < \lambda_2 - \lambda_1 < \pi$), then c is determined by solving the equation:

$$\lambda_2 - \lambda_1 = \mathrm{L}(c, 0, \pi) \text{ in the interval } 0 \le c \le 1.$$

the solution parameters are then given by:

$$s = \mathrm{A}(c, 0, \pi),\ \alpha_1 = \arcsin(c)\text{, and } \alpha_2 = \pi - \alpha_1.$$

This geodesic curve segment lies in the northern hemisphere. A second solution lies in the southern hemisphere in north-south symmetry.

Typical prograde case: This case assumes that $0 < \lambda_2 - \lambda_1 < \pi$ and $\varphi_2 \geq |\varphi_1|$, but not $\varphi_1 = \varphi_2 = 0$.

Define $c_{max} = \cos\varphi_2/\sqrt{1 - \varepsilon^2 \sin^2\varphi_2}$ and $\lambda_{\text{crit}} = \mathrm{L}(c_{max}, (\arcsin(\sin\varphi_1/\sin\varphi_2), \pi/2))$.

Then c may be determined by an iterative solution of the equation:

$$\lambda_2 - \lambda_1 = \mathrm{L}\big(c, \theta_1(c), \theta_2(c)\big) \text{ in the interval } 0 \leq c \leq c_{max}$$

where: θ_1, θ_2 and k are the following functions of c:

$$\theta_1(c) = \arcsin(\,\sin(\varphi_1)/k(c)\,),$$

$$\theta_2(c) = \begin{cases} \arcsin\big(\sin(\varphi_1)/k\,(c)\big), & \text{if } \lambda_2 - \lambda_1 < \lambda_{\text{crit}} \\ \pi/2, & \text{if } \lambda_2 - \lambda_1 = \lambda_{\text{crit}} \\ \pi - \arcsin\big(\sin(\varphi_1)/k\,(c)\big), & \text{if } \lambda_2 - \lambda_1 > \lambda_{\text{crit}} \end{cases} \text{, and}$$

$$k(c) = \sqrt{(1 - c^2)/(1 - c^2\varepsilon^2)}\,,$$

The solution parameters are determined by c:

$$s = \mathrm{A}\big(c, \theta_1(c), \theta_2(c)\big),$$

$$\alpha_1 = \operatorname{arctan2}\big(c\sqrt{1 - \varepsilon^2\sin^2\varphi_1}\,, \sqrt{1 - c^2}\cos\theta_1\big), \text{ and}$$

$$\alpha_2 = \operatorname{arctan2}\left(c\sqrt{1 - \varepsilon^2\sin^2\varphi_2}\,, \sqrt{1 - c^2}\cos\theta_2\right).$$

NOTE Extremely small values of c can cause numerical instability in some implementations.

Other prograde cases: If $0 < \lambda_2 - \lambda_1 < \pi$ and the cases above do not apply, a new pair of points $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$ that satisfy the typical prograde case constraints can be specified using parameters from the given pair $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. The indirect problem solution for points $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$, the shortest distance between them s, and the forward azimuths α_3 at $\boldsymbol{p}_3$ and α_4 at $\boldsymbol{p}_4$ will determine the solution for $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as follows:

If $|\varphi_2| \leq \varphi_1$, let $p_3 = (\lambda_1, \varphi_2)$ and $p_4 = (\lambda_2, \varphi_1)$. Then $\alpha_1 = \pi - \alpha_4$ and $\alpha_2 = \pi - \alpha_3$.

If $|\varphi_2| \leq -\varphi_1$, let $p_3 = (\lambda_1, -\varphi_2)$ and $p_4 = (\lambda_2, -\varphi_1)$. Then $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$.

If $|\varphi_1| \leq -\varphi_2$, let $p_3 = (\lambda_1, -\varphi_1)$ and $p_4 = (\lambda_2, -\varphi_2)$. Then $\alpha_1 = \pi - \alpha_3$ and $\alpha_2 = \pi - \alpha_4$.

In all these cases the arc length solution is the same, $s = \tilde{s}$, and the value of c and the multiplicity of shortest geodesic segments are also the same.

Retrograde cases: A retrograde case, $\lambda_2 < \lambda_1$, is converted to a prograde case with $p_3 = (\lambda_2, \varphi_1)$ and $p_4 = (\lambda_1, \varphi_2)$. Then $\alpha_1 = -\alpha_3$, $\alpha_2 = -\alpha_4$, and $s = \tilde{s}$. The value $(-c)$ from the prograde case is the retrograde solution value for c and the multiplicity of shortest geodesic segments are the same.

