6 Orientation – change of basis and rotation

6.1 Introduction

Orientation, change of basis operations, and rotation operations are closely related concepts that are important in many different application domains. Unfortunately, the terminology and notation used to describe these concepts is diverse, and often inconsistent, causing confusion and errors.

A change of basis operation inputs a position-vector expressed in one orthonormal frame and outputs a position-vector for the same position expressed in a different orthonormal frame. Change of basis operations are the foundation of coordinate conversion and transformation computations (see Clause 10). Change of basis operations are normatively defined in 6.2.

The orientation of a rigid spatial object describes its angular displacement, or attitude, with respect to a reference, and is part of its state, along with its position and other spatial characteristics. The orientation of one orthonormal frame with respect to a second orthonormal frame is the directed angular relationship between them, with the second orthonormal frame serving as the reference. The specification of an orientation is important in many application domains including graphical rendering, interpretation of imagery, analysis of directional sensor data, robotics, vehicle aspect tracking, and the computations of direction and trajectory. Orientation is normatively defined and related to both change of basis and rotation operations in 6.3.

A rotation operation inputs a position-vector expressed in an orthonormal frame and outputs a position-vector for a different position that is rotated about a specified axis by a specified angle, expressed in the same orthonormal frame. Rotation operations are critical to the representation of motion, force, and dynamics in many application domains, including mechanics, aviation, and astronomy. Rotation can be interpreted in terms of the physical movement of objects, abstract geometry, or mathematical operations including change of basis operations. Rotation operations are normatively defined in 6.4.

Change of basis and rotation operators are summarized in 6.5. Rotations and orientations are commonly expressed in various forms, including axis-angle, matrices, Euler angles, and quaternions. These forms are normatively defined in 6.6. Conversions between these forms are normatively defined in 6.7.

6.2 Change of basis

6.2.1 Introduction

Within a Euclidean vector space, change of basis operations allow a vector expressed in terms of a given basis to be re-expressed in terms of a different basis. Change of basis operations are used in many types of matrix computations. In this International Standard, change of basis operations are used to express position-vectors, directions, and vector quantities in terms of different orthonormal frames.

6.2.2 Change of basis operations

A *change of basis* operation acts on a position-vector expressed in one orthonormal frame and produces the equivalent position-vector expressed in terms of a different orthonormal frame. In general, a change of basis operation can include an angular component and, when the frame origins differ, a positional displacement component. In some contexts, a change of basis operation can also include a scaling component (see 7.3.2).

$\boldsymbol{E}$ and $\boldsymbol{F}$ are two right-handed 3D orthonormal frames with respective basis vectors specified as $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. There is interest in computing the coordinate of a position-vector provided in one frame in terms of the other frame. When the origins of the two frames are different, denote the respective frame origins by $\boldsymbol{O_E}$ and $\boldsymbol{O_F}$. The vector from the origin of frame $\boldsymbol{E}$ to the origin of frame $\boldsymbol{F}$ is $\overrightarrow{\boldsymbol{O_E O_F}}$, which is the origin of frame $\boldsymbol{F}$ expressed in terms of frame $\boldsymbol{E}$. The inverse vector from the origin of frame $\boldsymbol{F}$ to the origin of frame $\boldsymbol{E}$ is $\overrightarrow{\boldsymbol{O_F O_E}}$, which is the origin of frame $\boldsymbol{E}$ expressed in terms of frame $\boldsymbol{F}$.

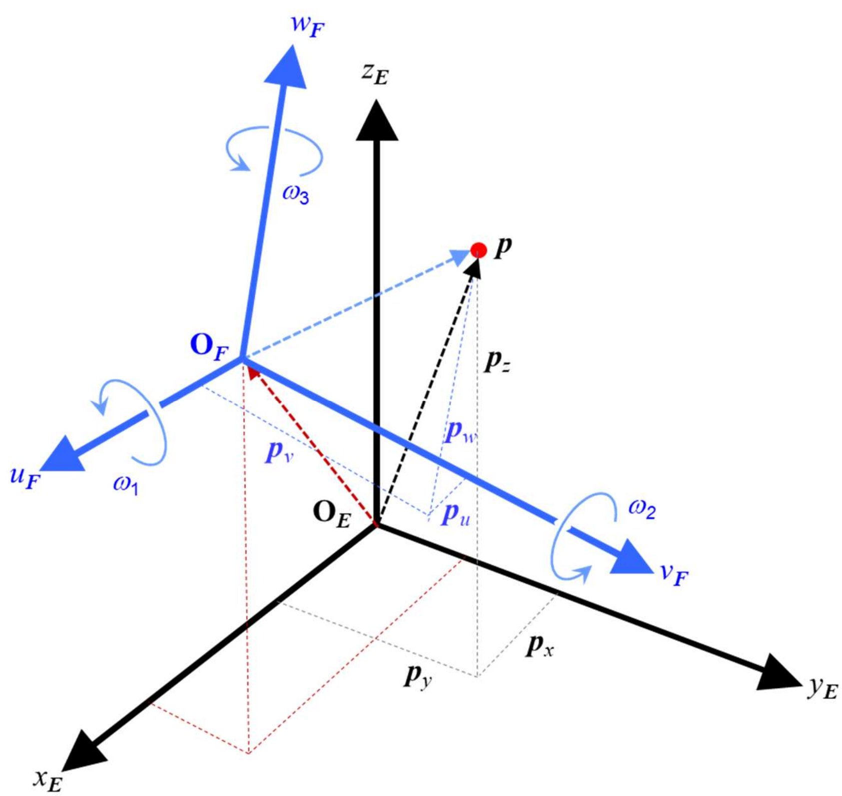

Figure 6.1 — Change of basis relationships

As Figure 6.1 illustrates, the position-vector $\boldsymbol{p}$ can be expressed with respect to the origin of frame $\boldsymbol{E}$ as $\overrightarrow{\boldsymbol{O_E}p}$, As Figure 6.1 further illustrates, $\boldsymbol{p}$ can also be expressed with respect to the origin of frame $\boldsymbol{E}$ as the vector sum $\overrightarrow{\boldsymbol{O_E}p} = \overrightarrow{\boldsymbol{O_E O_F}} + \overrightarrow{\boldsymbol{O_F}p}$. Thus, $\boldsymbol{p}$ can be expressed with respect to the origin of frame $\boldsymbol{F}$ as:

$$\overrightarrow{\boldsymbol{O_F p}}. = \overrightarrow{\boldsymbol{O_E p}} - \overrightarrow{\boldsymbol{O_E O_F}}$$

or, reversing the direction of $\overrightarrow{\boldsymbol{O_E O_F}}$:

$$\overrightarrow{\boldsymbol{O_F p}}. = \overrightarrow{\boldsymbol{O_F O_E}} + \overrightarrow{\boldsymbol{O_E p}}$$

The position-vector $\boldsymbol{p}$ represented in terms of frame $\boldsymbol{E}$ and denoted by $\boldsymbol{p_E}$, is the same vector as $\overrightarrow{\boldsymbol{O_E p}}$. Similarly, the position-vector $\boldsymbol{p}$ represented in terms of frame $\boldsymbol{F}$ and denoted by $\boldsymbol{p_F}$, is the same vector as $\overrightarrow{\boldsymbol{O_F}p}$. The transformation operation that re-expresses $\boldsymbol{p_E}$ in terms of frame $\boldsymbol{F}$ is:

$$\boldsymbol{p_F} = \overrightarrow{\boldsymbol{O_F O_E}} + \boldsymbol{\Omega_{F \leftarrow E} p_E},$$

where $\overrightarrow{\boldsymbol{O_F O_E}}$ denotes the positional displacement component and $\boldsymbol{\Omega_{F \leftarrow E}}$ denotes the angular displacement component. The direction of the positional displacement vector is from the origin of the target frame to the origin of the source frame. The inverse transformation operation that re-expresses $\boldsymbol{p_F}$ in terms of frame $\boldsymbol{E}$ is:

$$\boldsymbol{p_E} = \overrightarrow{\boldsymbol{O_E O_F}} + \boldsymbol{\Omega_{E \leftarrow F} p_F}.$$

If frames $\boldsymbol{E}$ and $\boldsymbol{F}$ have a common origin, there is no positional displacement component, and thus $\boldsymbol{p_E}$ can be re-expressed in terms of frame $\boldsymbol{F}$ using only the angular displacement component:

$$\boldsymbol{p_F} = \boldsymbol{\Omega_{F \leftarrow E}}\, \boldsymbol{p_E}.$$

The inverse transformation is:

$$\boldsymbol{p_E} = \boldsymbol{\Omega_{E \leftarrow F}}\, \boldsymbol{p_F}.$$

Throughout the remainder of this clause, unless otherwise specified, a common origin for both frames is assumed. Thus, the phrase *change of basis* is used to refer to only the angular displacement component of the operation, denoted by $\boldsymbol{\Omega}$ with appropriate subscripts.

For a position-vector $\boldsymbol{p}$, the frame $\boldsymbol{E}$ coordinate for $\boldsymbol{p}$ with respect to the common origin is $(p_x, p_y, p_z)_{\boldsymbol{E}}$, where each scalar value is the dot product of the position-vector with one of the basis vectors of the orthonormal frame:

$$p_x = \boldsymbol{p} \bullet \boldsymbol{x},\ p_y = \boldsymbol{p} \bullet \boldsymbol{y},\ p_z = \boldsymbol{p} \bullet \boldsymbol{z}.$$

Similarly, the frame $\boldsymbol{F}$ coordinate for $\boldsymbol{p}$ is $(p_u, p_v, p_w)_{\boldsymbol{F}}$, where

$$p_u = \boldsymbol{p} \bullet \boldsymbol{u},\ p_v = \boldsymbol{p} \bullet \boldsymbol{v},\ p_w = \boldsymbol{p} \bullet \boldsymbol{w}.$$

The linear combination with respect to frame $\boldsymbol{E}$ can be written as:

$$\boldsymbol{p} = p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}.$$

Using this expression for $\boldsymbol{p}$, the $\boldsymbol{F}$ frame coordinate components of $\boldsymbol{p}$ become:

$$\begin{aligned}
p_u &= \boldsymbol{p} \bullet \boldsymbol{u} = (p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}) \bullet \boldsymbol{u} = p_x \boldsymbol{x} \bullet \boldsymbol{u} + p_y \boldsymbol{y} \bullet \boldsymbol{u} + p_z \boldsymbol{z} \bullet \boldsymbol{u} \\
p_v &= \boldsymbol{p} \bullet \boldsymbol{v} = (p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}) \bullet \boldsymbol{v} = p_x \boldsymbol{x} \bullet \boldsymbol{v} + p_y \boldsymbol{y} \bullet \boldsymbol{v} + p_z \boldsymbol{z} \bullet \boldsymbol{v} \\
p_w &= \boldsymbol{p} \bullet \boldsymbol{w} = (p_x \boldsymbol{x} + p_y \boldsymbol{y} + p_z \boldsymbol{z}) \bullet \boldsymbol{w} = p_x \boldsymbol{x} \bullet \boldsymbol{w} + p_y \boldsymbol{y} \bullet \boldsymbol{w} + p_z \boldsymbol{z} \bullet \boldsymbol{w}
\end{aligned}$$

The matrix form of this system of linear equations is:

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}_{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{x} \bullet \boldsymbol{u} & \boldsymbol{y} \bullet \boldsymbol{u} & \boldsymbol{z} \bullet \boldsymbol{u} \\ \boldsymbol{x} \bullet \boldsymbol{v} & \boldsymbol{y} \bullet \boldsymbol{v} & \boldsymbol{z} \bullet \boldsymbol{v} \\ \boldsymbol{x} \bullet \boldsymbol{w} & \boldsymbol{y} \bullet \boldsymbol{w} & \boldsymbol{z} \bullet \boldsymbol{w} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}}, \text{ or}$$

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}_F = \boldsymbol{M}_{F \leftarrow E} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_E \text{, or}$$

$$\boldsymbol{p}_F = \boldsymbol{M}_{F \leftarrow E} \boldsymbol{p}_E$$

This matrix multiplication operation $\boldsymbol{M}_{F \leftarrow E}$ is equivalent to the change of basis operation $\boldsymbol{\Omega}_{F \leftarrow E}$:

$$\boldsymbol{p}_F = \boldsymbol{\Omega}_{F \leftarrow E}\, \boldsymbol{p}_E.$$

Since both frames are orthonormal and have a common origin, the relationship also represents the projection of the basis vectors of one frame onto the basis vectors of the other frame. The columns of $\boldsymbol{\Omega}_{F \leftarrow E}$ are the $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ basis vectors in terms of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ coordinate-components while the rows (or columns of the transpose matrix $\boldsymbol{\Omega}_{E \leftarrow F}$) are the $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ basis vectors in terms of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ coordinate-components.

$$\begin{gathered} \boldsymbol{\Omega}_{F \leftarrow E} = \boldsymbol{\Omega}_{E \leftarrow F}^{-1} = \begin{bmatrix} \boldsymbol{x} \bullet \boldsymbol{u} & \boldsymbol{y} \bullet \boldsymbol{u} & \boldsymbol{z} \bullet \boldsymbol{u} \\ \boldsymbol{x} \bullet \boldsymbol{v} & \boldsymbol{y} \bullet \boldsymbol{v} & \boldsymbol{z} \bullet \boldsymbol{v} \\ \boldsymbol{x} \bullet \boldsymbol{w} & \boldsymbol{y} \bullet \boldsymbol{w} & \boldsymbol{z} \bullet \boldsymbol{w} \end{bmatrix} = \boldsymbol{M}_{F \leftarrow E} \\ \boldsymbol{\Omega}_{E \leftarrow F} = \boldsymbol{\Omega}_{F \leftarrow E}^{-1} = \begin{bmatrix} \boldsymbol{u} \bullet \boldsymbol{x} & \boldsymbol{v} \bullet \boldsymbol{x} & \boldsymbol{w} \bullet \boldsymbol{x} \\ \boldsymbol{u} \bullet \boldsymbol{y} & \boldsymbol{v} \bullet \boldsymbol{y} & \boldsymbol{w} \bullet \boldsymbol{y} \\ \boldsymbol{u} \bullet \boldsymbol{z} & \boldsymbol{v} \bullet \boldsymbol{z} & \boldsymbol{w} \bullet \boldsymbol{z} \end{bmatrix} = \boldsymbol{M}_{E \leftarrow F} \end{gathered} \qquad (6.1)$$

These operators define the change of basis relationship between the two frames $\boldsymbol{E}$ and $\boldsymbol{F}$, allowing position-vector representations to be converted from one to the other, in either direction.

6.2.3 Direction cosine matrix

Expressing the basis vectors of one frame in terms of the other frame provides the relationship between the two frames. One way to express the relationship is based on the cosine of the angle between each basis vector of a frame and all basis vectors of the other frame. Since basis vectors are unit vectors, each dot product in Equation 6.1 is the cosine of the angle (θ) between the two indicated vectors (see A.2). A total of nine cosine values are required to describe the full relationship between two 3D frames. Arranged as a matrix, for frames $\boldsymbol{E}$ and $\boldsymbol{F}$ the nine cosine values are represented as:

$$\boldsymbol{\Omega}_{F \leftarrow E} = \boldsymbol{\Omega}_{E \leftarrow F}^{-1} = \begin{bmatrix} \cos(\boldsymbol{\theta}_{xu}) & \cos(\boldsymbol{\theta}_{yu}) & \cos(\boldsymbol{\theta}_{zu}) \\ \cos(\boldsymbol{\theta}_{xv}) & \cos(\boldsymbol{\theta}_{yv}) & \cos(\boldsymbol{\theta}_{zv}) \\ \cos(\boldsymbol{\theta}_{xw}) & \cos(\boldsymbol{\theta}_{yw}) & \cos(\boldsymbol{\theta}_{zw}) \end{bmatrix}$$

$$\boldsymbol{\Omega}_{E \leftarrow F} = \boldsymbol{\Omega}_{F \leftarrow E}^{-1} = \begin{bmatrix} \cos(\boldsymbol{\theta}_{ux}) & \cos(\boldsymbol{\theta}_{vx}) & \cos(\boldsymbol{\theta}_{wx}) \\ \cos(\boldsymbol{\theta}_{uy}) & \cos(\boldsymbol{\theta}_{vy}) & \cos(\boldsymbol{\theta}_{wy}) \\ \cos(\boldsymbol{\theta}_{uz}) & \cos(\boldsymbol{\theta}_{vz}) & \cos(\boldsymbol{\theta}_{wz}) \end{bmatrix}$$

The first matrix expresses the basis vectors of frame $\boldsymbol{E}$ in terms of frame $\boldsymbol{F}$. The columns of the matrix are the basis vectors of frame $\boldsymbol{E}$ in $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ coordinate components while the rows are the basis vectors of frame $\boldsymbol{F}$ in $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ coordinate components. It is noted that the sum of the square of the values in each column is one. The second matrix is the inverse and expresses the unit vectors of $\boldsymbol{F}$ in terms of $\boldsymbol{E}$. Each of these two matrices are often referred to as a *direction cosine matrix*.

6.2.4 Consecutive change of basis

Given three right-handed orthonormal frames $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ with a common origin and the change of basis operators $\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{E}}$ and $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and their inverses $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{D}}$ and $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$, the change of basis operators $\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{D}}$ are the compositions:

$$\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{F}} = \boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{E}} \circ \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$$

$$\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{D}} = \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}} \circ \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{D}}$$

This result generalizes to a chain of orthonormal frames with different origins. For a chain of length $\boldsymbol{N}$, denote the n^{th} frame by $\boldsymbol{F}_{\boldsymbol{n}}$, its origin by $\boldsymbol{O}_{\boldsymbol{n}}$ and the displacement vector from the F_j origin to the $F_{\boldsymbol{n}}$ origin by $\overrightarrow{\boldsymbol{O}_{\boldsymbol{j}}\boldsymbol{O}_{\boldsymbol{n}}}$ for $1 \le j, n \le N$. For any $1 \le j < n < N$, the change of basis operator, along with positional components, from frame $\boldsymbol{F}_{\boldsymbol{j}}$ to frame $\boldsymbol{F}_{\boldsymbol{n}}$ is denoted by $\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}} + \boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{j}}$.

For a chain of frames from $\boldsymbol{F}_{\boldsymbol{j}}$ to $\boldsymbol{F}_{\boldsymbol{n}}$, the composition of the consecutive chain of operations is:

$$\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}} + \boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{j}} = \left(\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{m}}} + \cdots + \overrightarrow{\boldsymbol{O}_{\boldsymbol{k}}\boldsymbol{O}_{\boldsymbol{j}}}\right) + \left(\boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{m}} \circ \ldots \circ \boldsymbol{\Omega}_{\boldsymbol{k}\leftarrow\boldsymbol{j}}\right) = \overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}} + \left(\boldsymbol{\Omega}_{\boldsymbol{n}\leftarrow\boldsymbol{m}} \circ \ldots \circ \boldsymbol{\Omega}_{\boldsymbol{k}\leftarrow\boldsymbol{j}}\right).,$$

since the vector sum of the chain of vectors $\left(\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{m}}} + \cdots + \overrightarrow{\boldsymbol{O}_{\boldsymbol{k}}\boldsymbol{O}_{\boldsymbol{j}}}\right)$ is equivalent to the single vector $\overrightarrow{\boldsymbol{O}_{\boldsymbol{n}}\boldsymbol{O}_{\boldsymbol{j}}}$.

6.2.5 Equivalence of change of basis and rotation operators

The common origin of the right-handed orthonormal frames $\boldsymbol{E}$ and $\boldsymbol{F}$ is a fixed point of the operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$. Euler's rotation theorem states that any length-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis that passes through the fixed point. This implies that $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ is equivalent to a rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ (see 6.4.2.1), where $\boldsymbol{n}$ is the axis of rotation passing through the origin and θ is the rotation angle. This operator rotates a position-vector $\boldsymbol{p}$ to $\boldsymbol{p}' = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{p})$. The equivalence of $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ is shown in A.12.

Applying $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of frame $\boldsymbol{E}$ yields the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ of frame $\boldsymbol{F}$:

$$\boldsymbol{u} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{x})$$

$$\boldsymbol{v} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{y})$$

$$\boldsymbol{w} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{z})$$

The rotation operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ can also be designated as $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$. Hence, the orientation of object-frame $\boldsymbol{F}$ with respect to reference-frame $\boldsymbol{E}$ is realized by both the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$. Thus, the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and the rotation operation $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$ are equivalent to each other:

$$\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}} = \boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$$

The difference between operators $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$ is in the interpretation of the output of the operation as either the change of basis for any position-vector in terms of the bases of $\boldsymbol{F}$ and $\boldsymbol{E}$ or as the rotation of that position vector about axis $\boldsymbol{n}$ though angle θ. Applying this rotation to each of the basis vectors of frame $\boldsymbol{E}$ yields the basis vectors of frame $\boldsymbol{F}$, in effect rotating frame $\boldsymbol{F}$ away from alignment with frame $\boldsymbol{E}$.

The direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and the rotation matrix that corresponds to the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$ are therefore also equivalent:

$$\begin{bmatrix} \boldsymbol{u}\bullet\boldsymbol{x} & \boldsymbol{v}\bullet\boldsymbol{x} & \boldsymbol{w}\bullet\boldsymbol{x} \\ \boldsymbol{u}\bullet\boldsymbol{y} & \boldsymbol{v}\bullet\boldsymbol{y} & \boldsymbol{w}\bullet\boldsymbol{y} \\ \boldsymbol{u}\bullet\boldsymbol{z} & \boldsymbol{v}\bullet\boldsymbol{z} & \boldsymbol{w}\bullet\boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \cos(\boldsymbol{\theta}_{\boldsymbol{ux}}) & \cos(\boldsymbol{\theta}_{\boldsymbol{vx}}) & \cos(\boldsymbol{\theta}_{\boldsymbol{wx}}) \\ \cos(\boldsymbol{\theta}_{\boldsymbol{uy}}) & \cos(\boldsymbol{\theta}_{\boldsymbol{vy}}) & \cos(\boldsymbol{\theta}_{\boldsymbol{wy}}) \\ \cos(\boldsymbol{\theta}_{\boldsymbol{uz}}) & \cos(\boldsymbol{\theta}_{\boldsymbol{vz}}) & \cos(\boldsymbol{\theta}_{\boldsymbol{wz}}) \end{bmatrix}$$

6.2.6 Change of basis and orientation

The change of basis operators $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ and $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ express the bidirectional angular relationship between the two orthonormal frames $\boldsymbol{E}$ and $\boldsymbol{F}$. This bidirectional relationship is expressed in terms of the angles between each pair of basis vectors in the direction cosine matrices. Thus, the orientation of orthonormal frame $\boldsymbol{F}$ with respect to orthonormal frame $\boldsymbol{E}$ is represented by the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$. Similarly, the orientation of orthonormal frame $\boldsymbol{E}$ with respect to orthonormal frame $\boldsymbol{F}$ is represented by the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$.

6.3 Orientation

6.3.1 Introduction

The orientation of a rigid object describes its angular displacement, or attitude, with respect to a reference. When the object is represented by an orthonormal frame attached to the object, the orientation of the object is represented by the angular displacement of the object's frame with respect to an orthonormal reference frame. This angular displacement can be specified in terms of either: 1) a change of basis that converts a coordinate from the object's frame to the reference frame, or 2) a rotation of the object's frame away from alignment with the reference frame.

Specification of and computations with orientations are defined with respect to orthonormal frames (see 5.2.3). An orthonormal frame serving in the role of an orientation reference is termed a *reference-frame*. An orthonormal frame that is, conceptually or physically, rigidly attached to an object of interest is termed an *object-frame*.

Object-frame attachment choices have significant effects on computational results and will affect interoperability if not clearly specified. An object-frame can be attached to an object in many ways. The choice of object-frame origin attachment point and alignment of axis directions is highly dependent on the application domain and is not addressed in this International Standard.

There are infinitely many ways to attach the origin of an orthonormal frame to an object. The origin can be located at any point within the spatial object of interest, at any point on its surface, or at any point nearby in space. Common selections include the centre of mass of the object, its geometric centre, a corner of the object (assuming it has corners), or its bounding volume such that the object is completely within the first octant.

Given a selected origin, there are infinitely many ways to orient the basis vectors of the orthonormal frame. If the object is a celestial body, the basis vectors might be aligned with its rotational axis, its magnetic field axis, or the direction of the closest star (such as the Sun). If the object is a vehicle, such as an aircraft, the basis vectors might be aligned based on its direction of forward motion or other common reference orientations. If the object is located on, or near, the surface of the Earth, common selections include east-north-up (ENU) and north-east-down (NED).

6.3.2 Orientation defined in terms of a change of basis operator

The *orientation* of an object-frame $\boldsymbol{F}$ with respect to a reference-frame $\boldsymbol{E}$ is equivalent to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ that converts a coordinate in object-frame $\boldsymbol{F}$ to a corresponding coordinate in reference-frame $\boldsymbol{E}$.

6.3.3 Orientation defined in terms of a rotation operator

The *orientation* of an object-frame $\boldsymbol{F}$ with respect to a reference-frame $\boldsymbol{E}$ is also equivalent to the origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$ (see 6.4.3.5) that rotates the object-frame $\boldsymbol{F}$ away from alignment with the reference-frame $\boldsymbol{E}$.

6.3.4 Orientation Contexts

The designation of reference-frame and object-frame is context dependent. A frame may be associated with an object that operates within the object-space of another object that acts as a reference. Applications may need to relate the positions, orientations, and other properties of two or more objects of interest to one another, either by choosing one of them as the reference, or by choosing a separate object as the reference.

There are also use cases for which several reference-frames are used. In such cases, it may be necessary or desirable to express an object's orientation with respect to multiple reference-frames.

If the reference-frames are independent of one another, the orientation of the object-frame with respect to each of the reference-frames must be separately determined. If the relationships between the reference-frames are known, and the orientation of the object-frame with respect to at least one reference-frame is known, its relationship to the other reference-frames can be derived.

A sequence of orthonormal frames for a set of objects may form a chain. The first frame in the sequence is an object-frame. Subsequent frames are each the reference-frame for its preceding frame and also the object-frame for its succeeding frame in the sequence[1].

In dynamic applications, the origin and basis vectors that specify an object-frame and/or reference-frame may be functions of time or a time-stamped sequence of frames. Thus, a robotic arm may be modelled as a frame chain with an object-frame for the hand segment as the start of the chain and orthonormal frames for each jointed segment in sequential order away from the hand. As the mechanical assembly moves, each frame's orientation values change as a function of time. Given a specific time, each frame has fixed orientation values with respect to its respective reference-frame.

6.4 Rotation

6.4.1 Introduction

A rotation operation rotates one or more points about a given directed axis of rotation through an angle θ.

[1] Some applications reverse the order of the sequence.

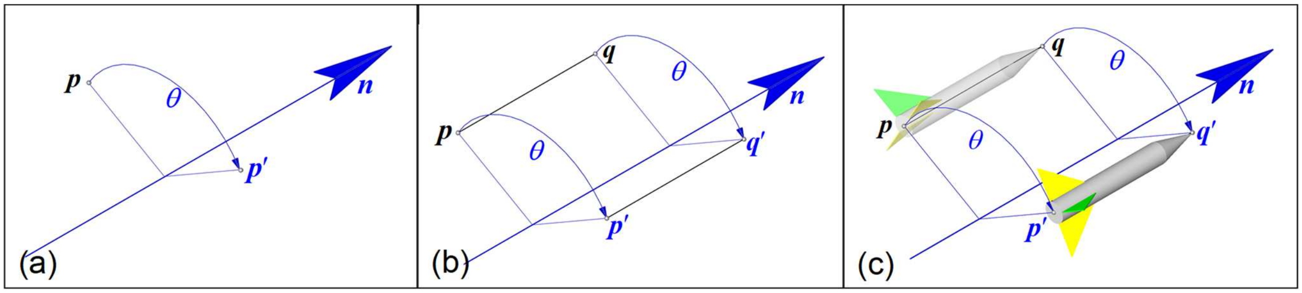

Figure 6.2 — Rotation operator applied to a point, a line segment, and a 3D object

As shown in Figure 6.2(a), the rotation operation can be applied to a single point $\boldsymbol{p}$, producing the rotated point $\boldsymbol{p'}$. As shown in Figure 6.2(b), the same rotation operation can be applied to any geometric primitive, such as the line segment with endpoints $\boldsymbol{p}$ and $\boldsymbol{q}$, producing the rotated line segment defined by the rotated endpoints $\boldsymbol{p'}$ and $\boldsymbol{q'}$. Furthermore, as shown in Figure 6.2(c), this rotation operation can be applied to any rigid three-dimensional object. All 3D objects are assumed to be rigid bodies. Each of the infinite set of points that make up the object is rotated about the axis in the same manner.

Rotation operator concepts and various mathematical representations of rotations have been in wide use from before the time of Euler's work on the subject. As a result, there are many different treatments in the literature, using similar terms with different meanings and different notational conventions. For this reason, rotation terms and notation used in this International Standard are fully defined in this clause. Different mathematical representations of rotations have various application-specific advantages including data storage size, computational efficiencies and/or direct use of measurement data. Various representations of rotations in common use are given in 6.6 and methods of converting between representations are specified in 6.7.

6.4.2 Coordinate-free rotation

6.4.2.1 Origin-fixed rotation

Euler's rotation theorem states that any length-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis that passes through the fixed point. A rotation about an axis that passes through a designated origin is termed an *origin-fixed rotation* since the origin is a fixed point of the rotation. An origin-fixed rotation is a coordinate system-independent (i.e., coordinate-free) operation, since only an origin is required, rather than a complete orthonormal basis.

An origin-fixed rotation, denoted $\boldsymbol{R_n}\langle\theta\rangle$, is specified by a directed axis that is the span of a unit vector $\boldsymbol{n}$ and a signed rotation angle θ. The rotation direction is determined by the *right-hand rule*: conceptually, if the right hand holds the axis with thumb pointing in the direction of the vector $\boldsymbol{n}$, the fingers curl in the positive angle direction (increasing θ). The rotation angle θ is measured from the starting position of a vector $\boldsymbol{r}$ to its rotated position $\boldsymbol{r}' = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{r})$. Large rotations (greater than one full revolution) are important in some applications, however, in this standard, angles shall be considered equivalent modulo 2π.

Rotation about an arbitrary axis in space can be treated as equivalent to an origin-fixed rotation, as the axis can be translated to the origin before the rotation operation and translated back after the rotation operation. Thus, rotation operations are translation independent. In the remainder of this clause, all rotation operations are assumed to be origin-fixed rotation operations, unless otherwise indicated.

6.4.2.2 Rodrigues' rotation formula

The action performed by an origin-fixed rotation $\boldsymbol{R_n}\langle\theta\rangle$ on an arbitrary position-vector $\boldsymbol{p}$ may be computed using *Rodrigues' rotation formula* (see [BERN]):

$$\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{p}) = \cos(\theta)\,\boldsymbol{p} + (1-\cos(\theta))(\boldsymbol{p} \bullet \boldsymbol{n})\boldsymbol{n} + \sin(\theta)\,\boldsymbol{n} \times \boldsymbol{p} \tag{6.2}$$

Using Lagrange's formula, $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{r}) = (\boldsymbol{r} \bullet \boldsymbol{n})\boldsymbol{n} - (\boldsymbol{n} \bullet \boldsymbol{n})\boldsymbol{r}$ with $(\boldsymbol{n} \bullet \boldsymbol{n}) = 1$, the formula terms may be rearranged to this useful alternate form:

$$\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{p}) = \boldsymbol{p} + (1-\cos(\theta))\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{p}) + \sin(\theta)\,\boldsymbol{n} \times \boldsymbol{p}$$

As is evident from the sine and cosine terms, $\boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta + k2\pi\rangle$, where k is any positive or negative integer value.

NOTE Rodrigues' rotation formula is a coordinate-free specification of the action of a rotation operator on a position-vector. That is to say, the formula does not use coordinate components from any basis for the position-vector terms appearing in the formulation. (See A.2 Notes 2 and 3 for coordinate-free expressions of the vector dot and cross products.)

6.4.2.3 Rotation properties

An origin-fixed rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ is linear and length-preserving (see A.2). That is, given a scalar α and any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the following hold:

$\boldsymbol{R_n}\langle\theta\rangle(a\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{u}) + \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{v})$	linearity
$\lVert\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{v}) - \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{u})\rVert = \lVert\boldsymbol{v} - \boldsymbol{u}\rVert$	length-preserving.

Since the Pythagorean theorem holds in Euclidean space, the length-preserving property implies the angle preserving property that an angle between two vectors is preserved when they are rotated together. The composition of two origin-fixed rotation operators is not commutative unless the two axes of rotation are co-linear. The composition of three or more origin-fixed rotation operators is associative.

An origin-fixed rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ is invertible with inverse $\boldsymbol{R_n^{-1}}\langle\theta\rangle = \boldsymbol{R_n}\langle-\theta\rangle = \boldsymbol{R_{-n}}\langle\theta\rangle$. The expression $-\theta$ denotes the angle of rotation that is the additive inverse of the signed quantity θ, and $-\boldsymbol{n}$ denotes that the direction of the rotation axis is the reverse of the axis spanned by $\boldsymbol{n}$. $\boldsymbol{R_n}\langle 0\rangle = \boldsymbol{I}$ the identity operator.

NOTE The fact that there are multiple ways to invert a rotation operator $\boldsymbol{R_n}\langle\theta\rangle$, i.e., by reversing the sign of the rotation angle or by reversing the direction of the rotation axis, is a common source of confusion and errors with working with rotation operations. In this International Standard, $\boldsymbol{R_n^{-1}}\langle\theta\rangle$ is used to denote the inverse unless it is necessary to specify the manner in which the operator is being inverted.

6.4.2.4 Consecutive rotations

In some applications, two or more consecutive rotation operations are used to produce a desired end state for an object of interest. This sequence of rotation operations is a functional composition (see A.5) and is equivalent to a single rotation operation. However, there are two issues that arise when consecutive rotation operations are performed: 1) the end state of the object depends on the order in which the rotation operations are performed; and 2) the axis of rotation for the second and any subsequent rotation operations may or may not be modified by previous rotation operations.

In general, the composition of rotation operators is not commutative. That is, given two origin-fixed rotation operators, $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{R_m}\langle\varphi\rangle$, applied sequentially to a rigid body representing an object of interest:

$$\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle \neq \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle.$$

Consecutive rotations are commutative only in the special case that the two rotation axes are co-linear, that is when $\boldsymbol{m} = \pm\boldsymbol{n}$:

$$\boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle = \boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta \pm \varphi\rangle, \text{ when } \boldsymbol{m} = \pm\boldsymbol{n}, \text{ with matching signs.}$$

The composition of rotation operators is associative:

$$\boldsymbol{R_k}\langle\gamma\rangle \circ (\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle) = (\boldsymbol{R_k}\langle\gamma\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle) \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_k}\langle\gamma\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle.$$

Example. Figures 6.3 and 6.4 illustrate two different sequences of consecutive rotations of a rigid body. The different sequences of consecutive rotations produce different end states. Two consecutive rotation operations, $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{R_m}\langle\varphi\rangle$ are applied to the same 3D object. In this example, the rotation axis $\boldsymbol{n}$ is parallel to the long axis of the object, and the rotation axis $\boldsymbol{m}$ is perpendicular to the axis $\boldsymbol{n}$. However, these conditions are not significant. The only relevant constraint on the two axes is that they are not co-linear.

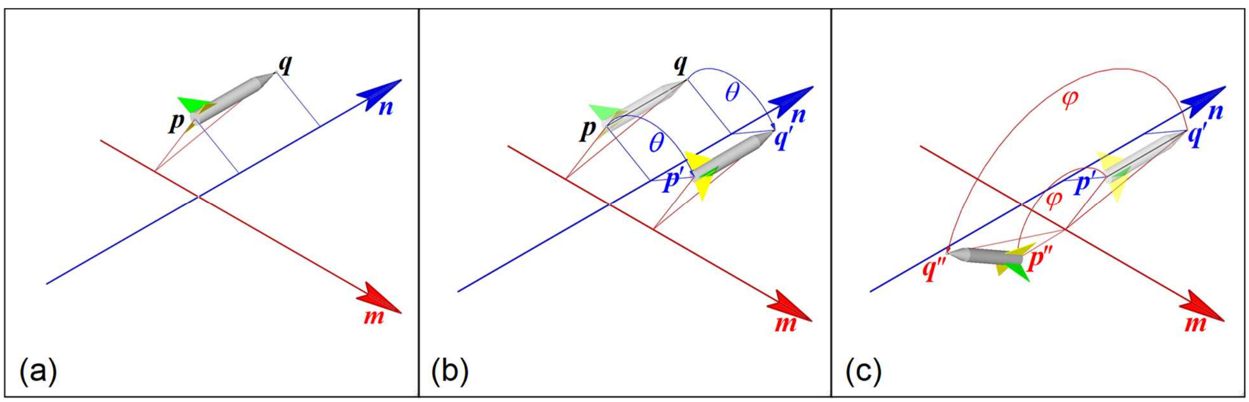

Figure 6.3 — Consecutive rotation operations: axis $\boldsymbol{n}$ followed by axis $\boldsymbol{m}$

In Figure 6.3, the first rotation operation performed is $\boldsymbol{R_n}\langle\theta\rangle$, and the second rotation operation is $\boldsymbol{R_m}\langle\varphi\rangle$. Figure 6.3(a) shows the initial configuration, Figure 6.3(b) shows the result of the first rotation operation $\boldsymbol{R_n}\langle\theta\rangle$, and Figure 6.3(c) shows the result of the second rotation operation $\boldsymbol{R_m}\langle\varphi\rangle$. Using composition notation in right-to-left order, this can be written as $\boldsymbol{R_m}\langle\boldsymbol{\varphi}\rangle \circ \boldsymbol{R_n}\langle\boldsymbol{\theta}\rangle$.

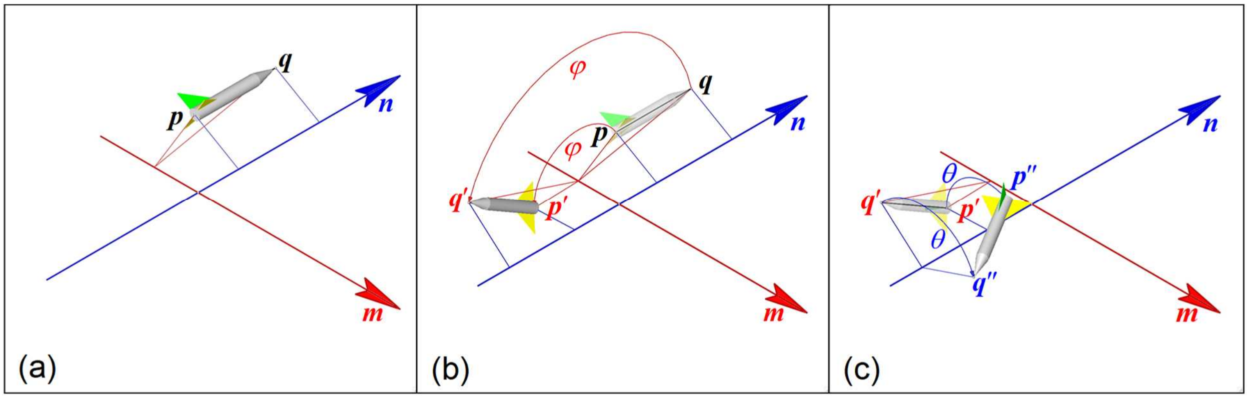

Figure 6.4 — Consecutive rotation operations: axis $\boldsymbol{m}$ followed by axis $\boldsymbol{n}$

In Figure 6.4, the order of the rotation operations is reversed. The first rotation operation performed is $\boldsymbol{R_m}\langle\varphi\rangle$, and the second rotation operation performed is $\boldsymbol{R_n}\langle\theta\rangle$. Figure 6.4(a) shows the initial configuration, Figure 6.4(b) shows the result of the first rotation operation $\boldsymbol{R_m}\langle\varphi\rangle$, and Figure 6.4(c) shows the result of the second rotation operation $\boldsymbol{R_n}\langle\theta\rangle$. Using composition notation in right-to-left order, this can be written as $\boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle$. The end states of the 3D object in Figures 6.3(c) and 6.4(c) are different from each other.

Thus far, the rotation axes $\boldsymbol{n}$ and $\boldsymbol{m}$ were assumed to remain fixed in space. However, if the axes are attached to the object of interest, they will be rotated with it.

The terms *space-fixed* convention and *body-fixed* convention distinguish between the case where the axes remain fixed in space and the case where the axes rotate with the object . Other terminology used for the space-fixed and body-fixed cases include extrinsic and intrinsic rotations, and fixed-frame and moving-frame rotations. Confusion between the space-fixed and body-fixed conventions is a common source of errors when working with rotation operations.

In the space-fixed convention the axes $\boldsymbol{n}$ and $\boldsymbol{m}$ remain stationary, and the rotations are applied only to the object. The resulting composite operation, in right-to-left operator composition order, is given by:

$$\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle \qquad \text{space-fixed convention.}$$

In the body-fixed convention, the rotations are also applied to the axes $\boldsymbol{n}$ and $\boldsymbol{m}$, so that the axes rotate together with the object. The first rotation $\boldsymbol{R_n}\langle\theta\rangle$ does not affect the axis $\boldsymbol{n}$, $\boldsymbol{n} = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{n})$, but rotates axis $\boldsymbol{m}$ to a new state $\boldsymbol{m'}$, $\boldsymbol{m'} = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{m})$. The second rotation in this convention uses the rotation axis in its new state $\boldsymbol{m'}$. The resulting composite operation, in right-to-left operator composition order, is given by:

$$\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle \qquad \text{body-fixed convention.}$$

In typical applications, the axis $\boldsymbol{m}$ is known, but additional computation would be required to determine $\boldsymbol{m'}$. However, it can be shown (see A.11) that reversing the order of the rotation operations in the space-fixed convention is the equivalent of the body-fixed convention:

$$\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle. \tag{6.3}$$

The term *space-fixed equivalent of body-fixed* convention is used for this method of reversing the order of rotations in the space-fixed convention to achieve the equivalent result to the body-fixed convention.

Thus, the three cases can be expressed as:

$\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle$ space-fixed convention, and

$\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle$ body-fixed convention

$\boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle$ space-fixed equivalent of body-fixed convention

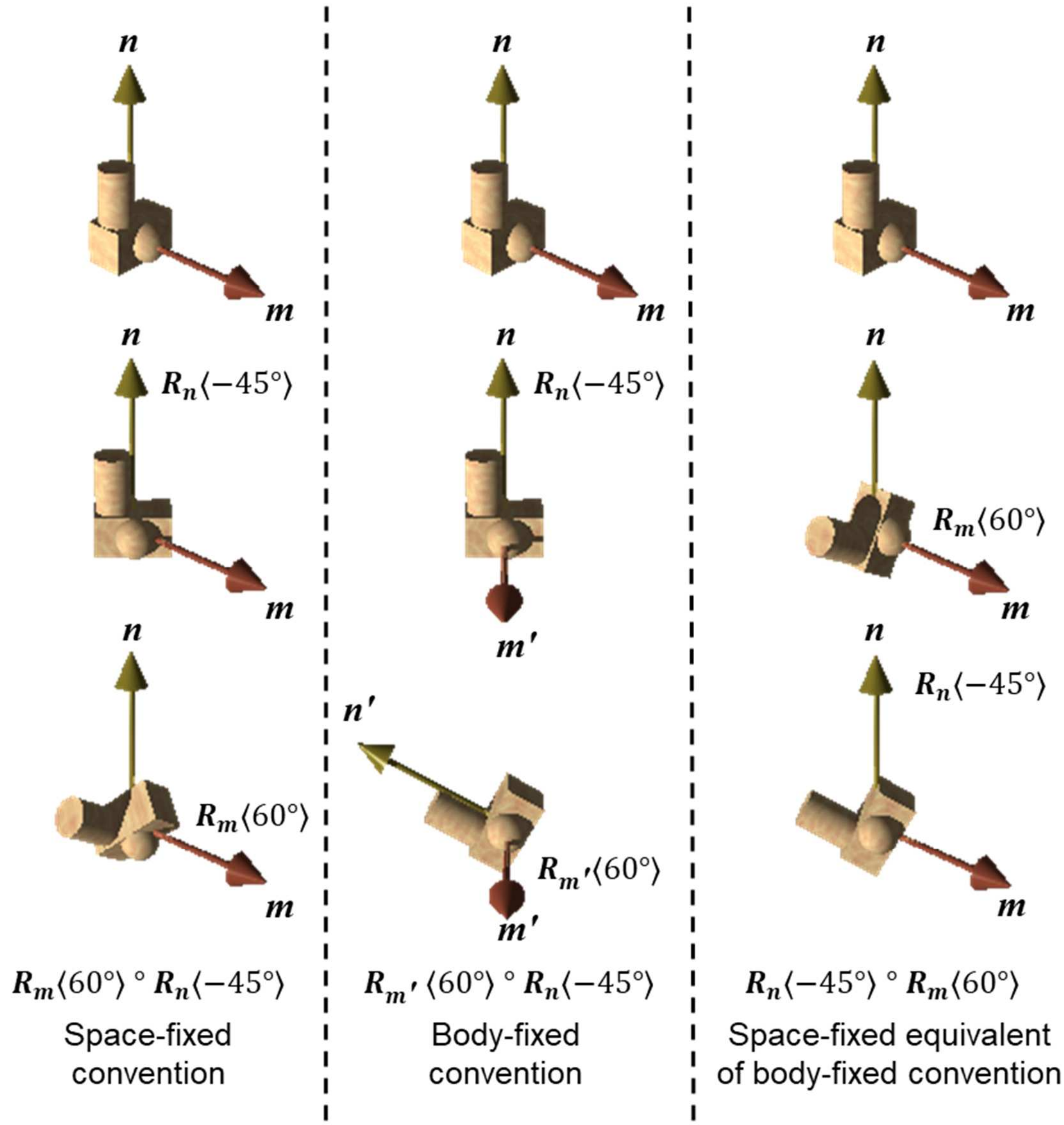

Figure 6.5 — Space-fixed, body-fixed, and space-fixed equivalent of body-fixed conventions

Figure 6.5 shows two consecutive rotations of a complex wooden block in the space-fixed, body-fixed, and space-fixed equivalent of body-fixed conventions.

Example. Figures 6.6 through 6.8 illustrate the difference between the space-fixed and body-fixed conventions. Starting with the same initial configuration shown in Figures 6.6(a), 6.7(a), and 6.8(a), two consecutive origin-fixed rotation operations, $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{R_m}\langle\varphi\rangle$ are applied to the same 3D object in different orders.

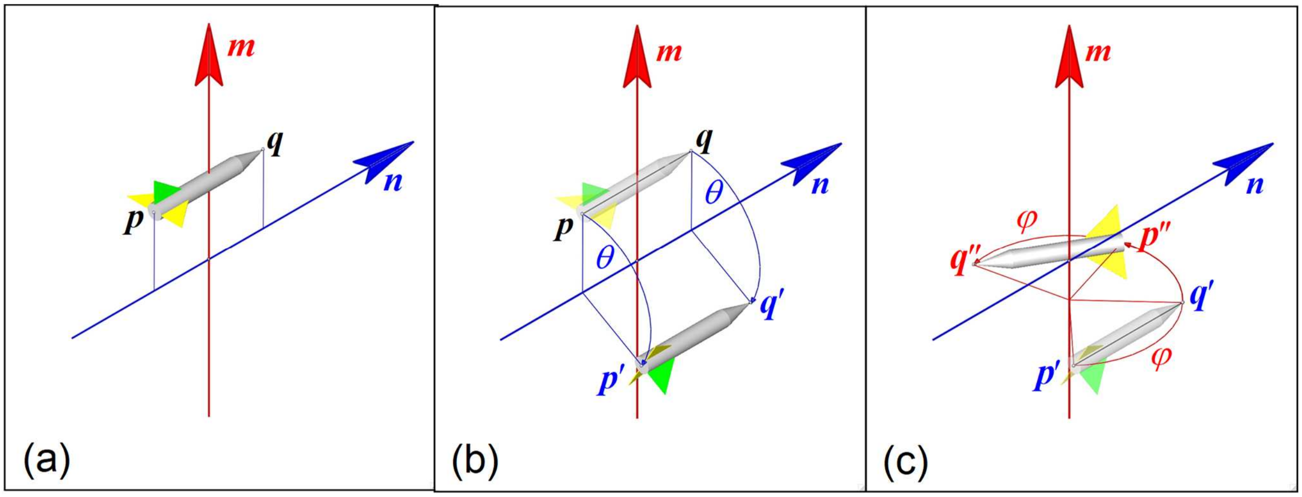

Figure 6.6 — Space-fixed convention

Figure 6.6 illustrates the space-fixed convention. Figure 6.6(b) shows the result of the first rotation operation $\boldsymbol{R_n}\langle\theta\rangle$, using the space-fixed convention. The rotated points are labeled $\boldsymbol{p'}$ and $\boldsymbol{q'}$ to distinguish them from the original states of the points $\boldsymbol{p}$ and $\boldsymbol{q}$. The rotation operation is applied only to the object. Figure 6.6(c) shows the result of the second rotation operation $\boldsymbol{R_m}\langle\varphi\rangle$. The points $\boldsymbol{p}$" and $\boldsymbol{q}$" are the final states of the original $\boldsymbol{p}$ and $\boldsymbol{q}$.

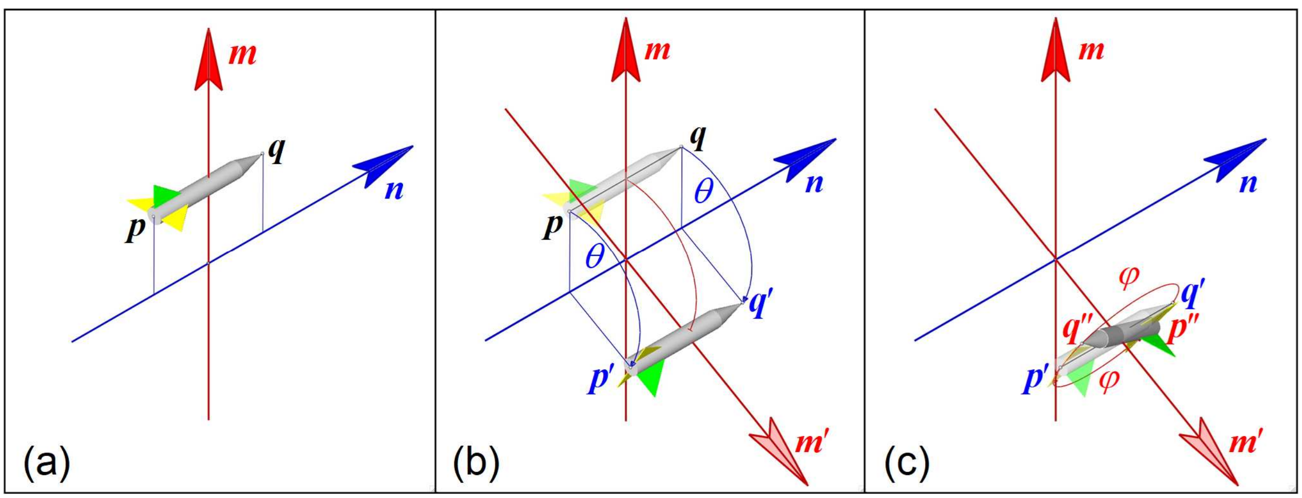

Figure 6.7 — Body-fixed convention

Figure 6.7 illustrates the body-fixed convention. Figure 6.7(b) shows the result of the first rotation operation $\boldsymbol{R_n}\langle\theta\rangle$, using the body-fixed convention. The rotation operation is applied to the axis $\boldsymbol{m}$ as well as to the object. The resulting rotated axis is labelled $\boldsymbol{m'}$ to distinguish it from the original state of axis $\boldsymbol{m}$. Figure 6.7(c) shows the result of the second rotation operation $\boldsymbol{R_{m'}}\langle\varphi\rangle$. In general, the body-fixed convention results in a different final state of the object than the space-fixed convention.

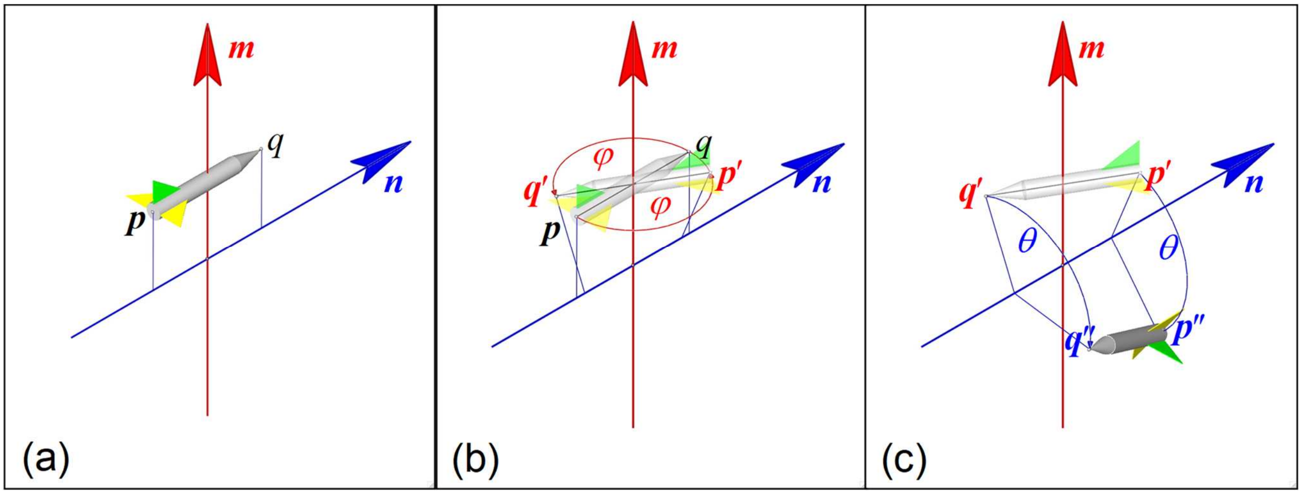

Figure 6.8 — Space-fixed equivalent of body-fixed convention

Figure 6.8 illustrates the result of applying the two rotation operations in the space-fixed equivalent of body-fixed convention. Figure 6.8(b) shows the result of the first rotation operation $\boldsymbol{R_m}\langle\varphi\rangle$. The rotation operation is applied only to the object. Figure 6.8(c) shows the result of the second rotation operation $\boldsymbol{R_n}\langle\theta\rangle$. Although the intermediate states shown in Figures 6.7(b) and 6.8(b) are different, the final states of the object shown in Figures 6.7(c) and 6.8(c) are identical, illustrating that reversing the order of the space-fixed convention rotation operations is equivalent to the body-fixed convention.

The equivalence expressed in Equation 6.3 may be generalized to more than two rotation operators. Given a third origin-fixed rotation $\boldsymbol{R_k}\langle\gamma\rangle$, let $\boldsymbol{k'} = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{k})$ and let $\boldsymbol{k}" = \boldsymbol{R_{m'}}\langle\varphi\rangle(\boldsymbol{k'}) = \boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{k}) = \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle(\boldsymbol{k})$, then:

$$\boldsymbol{R_{k"}}\langle\gamma\rangle \circ (\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle) = (\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle) \circ \boldsymbol{R_k}\langle\gamma\rangle = (\boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle) \circ \boldsymbol{R_k}\langle\gamma\rangle, \text{ or:}$$
$$\boldsymbol{R_{k"}}\langle\gamma\rangle \circ \boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_k}\langle\gamma\rangle \qquad \text{body-fixed convention.} \tag{6.4}$$

And the two generalized cases can be expressed as:

$\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_k}\langle\gamma\rangle$ space-fixed convention, and

$\boldsymbol{R_k}\langle\gamma\rangle \circ \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle$ space-fixed equivalent of body-fixed convention.

6.4.3 Frame-based rotation

6.4.3.1 Introduction

In 6.4.2, rotation operators were defined without requiring a coordinate-system or orthonormal frame, specifying only an origin. Rotations were treated as coordinate-frame-independent operations. However, to perform computations on the coordinates of position-vectors in rotation operations, it is necessary to choose an orthonormal frame by specifying a set of basis vectors (see 5.2.3).

Designating an orthonormal frame based at a given origin allows position-vectors to be represented by coordinate tuples and allows linear operations to be represented by matrix multiplications of coordinate tuples. This reduction to tuples and matrices is important in many application domains.

The terms *frame* and *orthonormal frame* are used interchangeably to denote a right-handed orthonormal frame (see 5.2.3). The concepts associated with frame-based rotations include both rotation of vectors within a frame and rotation of the frame itself.

6.4.3.2 Rotation of position-vectors

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, a position-vector $\boldsymbol{p}$ is represented by the scalar triple $(p_x, p_y, p_z)_{\boldsymbol{E}}$ where the scalars satisfy the equation $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$. Since $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ is an orthonormal basis, the solution is unique and is given by: $p_x = \boldsymbol{p} \bullet \boldsymbol{x}$, $p_y = \boldsymbol{p} \bullet \boldsymbol{y}$, $p_z = \boldsymbol{p} \bullet \boldsymbol{z}$. Thus, for any position-vector $\boldsymbol{p}$ and orthonormal basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$:

$$\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = (\boldsymbol{p} \bullet \boldsymbol{x})\boldsymbol{x} + (\boldsymbol{p} \bullet \boldsymbol{y})\boldsymbol{y} + (\boldsymbol{p} \bullet \boldsymbol{z})\boldsymbol{z}. \tag{6.5}$$

Using Equation 6.5, the result of any linear operator $\boldsymbol{L}$ acting on an arbitrary position-vector $\boldsymbol{p}$ is completely determined by the three values that the operator assigns to the basis position vectors:

$$\boldsymbol{L}(\boldsymbol{p}) = p_x\boldsymbol{L}(\boldsymbol{x}) + p_y\boldsymbol{L}(\boldsymbol{y}) + p_z\boldsymbol{L}(\boldsymbol{z}).$$

Thus, any linear operator may be represented as a matrix multiplication of coordinates. Coordinates are also necessary for other representations of rotation operations (see 6.6) and are otherwise important in many application domains.

The notation $[\boldsymbol{L}]_{\boldsymbol{E}}$ shall denote the matrix representation of the linear operator $\boldsymbol{L}$ operating by matrix multiplication of coordinates in orthonormal frame $\boldsymbol{E}$. The notional subscript is omitted when the relevant frame is otherwise indicated.

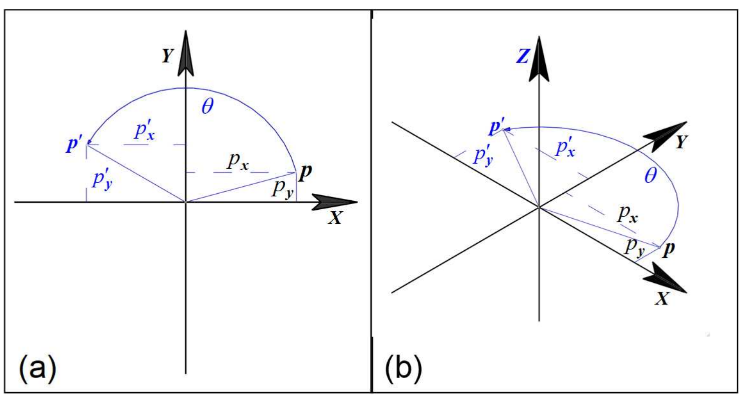

Figure 6.9 — Rotation of a position-vector within an orthonormal frame

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, the origin-fixed rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ applied to a position-vector $\boldsymbol{p}$ produces a rotated position-vector $\boldsymbol{p}'$. The rotated position-vector $\boldsymbol{p}'$ has coordinates $[p_x', p_y', p_z']_{\boldsymbol{E}}$ in frame $\boldsymbol{E}$. Figure 6.9(a) illustrates the top view of a rotation of the point $\boldsymbol{p}$ through an angle θ about the z-axis (which points out of the page), yielding the rotated point $\boldsymbol{p}'$. Figure 6.9(b) shows an isometric view of the same rotation operation.

The rotation operation $\boldsymbol{R_n}\langle\theta\rangle$, where the rotation axis $\boldsymbol{n}$ has coordinates $[n_x, n_y, n_z]_{\boldsymbol{E}}$ in the orthonormal frame $\boldsymbol{E}$, can be represented as a rotation matrix multiplication $\boldsymbol{p}_{\boldsymbol{E}}' = [\boldsymbol{R_n}\langle\theta\rangle]_{\boldsymbol{E}}\boldsymbol{p}_{\boldsymbol{E}}$. The matrix form of Rodrigues' rotation formula (see Equation 6.2) is:

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] = [\boldsymbol{I}_{3\times3} + \sin(\theta)\,\boldsymbol{S}_{\boldsymbol{n}} + (1-\cos(\theta))\boldsymbol{S}_{\boldsymbol{n}}^2]$$
$$= [\cos(\theta)\,\boldsymbol{I}_{3\times3} + (1-\cos(\theta))\boldsymbol{n}\otimes\boldsymbol{n} + \sin(\theta)\,\boldsymbol{S}_{\boldsymbol{n}}]$$

where:

$\boldsymbol{S}_{\boldsymbol{n}} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$ is the skew-symmetric matrix associated with $\boldsymbol{n} = [n_x \quad n_y \quad n_z]^{\mathrm{T}}$ and

$\boldsymbol{n}\otimes\boldsymbol{n} = \begin{bmatrix} n_x n_x & n_x n_y & n_x n_z \\ n_y n_x & n_y n_y & n_y n_z \\ n_z n_x & n_z n_y & n_z n_z \end{bmatrix}$ is the outer-product of $\boldsymbol{n}$ with itself.

Expanding the matrix elements yields:

$$\begin{aligned}[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] &= \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} n_x n_x & n_x n_y & n_x n_z \\ n_y n_x & n_y n_y & n_y n_z \\ n_z n_x & n_z n_y & n_z n_z \end{bmatrix} \\ &\quad + \sin\theta \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z\sin\theta & (1-\cos\theta)n_x n_z + n_y\, sin\,\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x\, sin\,\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta & (1-\cos\theta)n_z n_y + n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix}\end{aligned} \tag{6.6}$$

NOTE In general, the matrix representation of a linear operator such as $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ depends on the selection of a basis. In the case of Equation 6.6 the matrix coefficient values depend on the coordinate-component values of the rotation axis $\boldsymbol{n}$. The coordinate-component values of the rotation axis $\boldsymbol{n}$ depend on the basis vectors of the orthonormal frame $\boldsymbol{E}$. In a different frame, the rotation axis $\boldsymbol{n}$ would have different coordinate-component values, resulting in a different matrix.

6.4.3.3 Rotation of basis vectors and orthonormal frames

The origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ can be applied to each of the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the orthonormal frame $\boldsymbol{E}$ in the same manner as to any other position-vector:

$$u = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{x}),$$
$$v = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{y}),$$
$$w = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{z}).$$

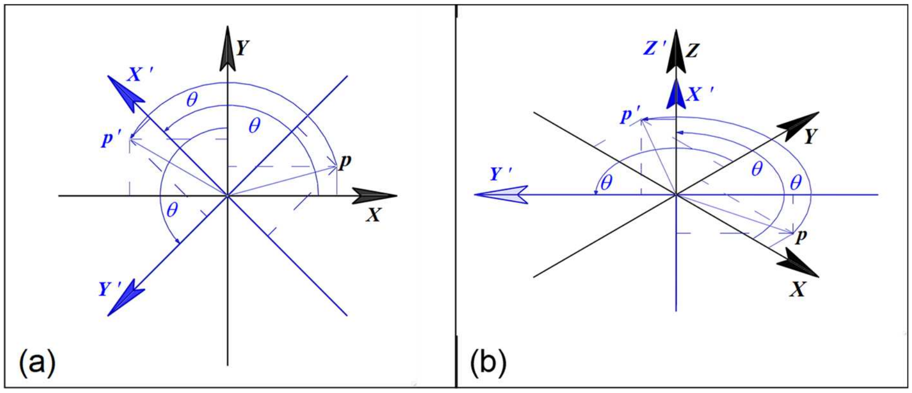

Figure 6.10 — Rotation of the basis vectors in the same direction as the position-vector

Figure 6.10(a) and 6.10(b) are top and isometric views that illustrate the $\boldsymbol{x}$- and $\boldsymbol{y}$-axes of the orthonormal frame $\boldsymbol{E}$ being rotated about the z-axis through the same angle θ, in the same direction, as the position-vector $\boldsymbol{p}$, yielding the rotated basis vectors $u = \boldsymbol{R_z}\langle\theta\rangle(\boldsymbol{x})$, $v = \boldsymbol{R_z}\langle\theta\rangle(\boldsymbol{y})$, and $w = \boldsymbol{z} = \boldsymbol{R_z}\langle\theta\rangle(\boldsymbol{z})$.

Although the rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ operates on individual vectors, when applied to the set of basis vectors of a frame $\boldsymbol{E}$, the result can be conceptually considered to have rotated that frame, resulting in a new frame denoted $\boldsymbol{F}$. When used in this way, the operation denoted by $\boldsymbol{R_n}\langle\theta\rangle$ rotates frame $\boldsymbol{F}$ away from frame $\boldsymbol{E}$, and therefore is also denoted by $\boldsymbol{R_{E\to F}}$. As illustrated in Figure 6.10, although the positions of $\boldsymbol{p}$ and $\boldsymbol{p'}$ are different, the coordinate of $\boldsymbol{p'}$ in frame $\boldsymbol{F}$, $\left(p'_{\boldsymbol{u}}, p'_{\boldsymbol{v}}, p'_{\boldsymbol{w}}\right)_{\boldsymbol{F}}$, has coordinate-component values that are identical to the coordinate-component values of $\boldsymbol{p}$ in frame $\boldsymbol{E}$, $\left(p_{\boldsymbol{x}}, p_{\boldsymbol{y}}, p_{\boldsymbol{z}}\right)_{\boldsymbol{E}}$:

$$
\begin{aligned}
&p'_{\boldsymbol{u}} = p_{\boldsymbol{x}},\\
&p'_{\boldsymbol{v}} = p_{\boldsymbol{y}},\\
&p'_{\boldsymbol{w}} = p_{\boldsymbol{z}},\\
&\boldsymbol{p'} = p'_{\boldsymbol{u}}\boldsymbol{u} + p'_{\boldsymbol{v}}\boldsymbol{v} + p'_{\boldsymbol{w}}\boldsymbol{w}.
\end{aligned}
$$

6.4.3.4 Principal rotations

Principal rotations are defined with respect to a given orthonormal frame. Each basis vector $\boldsymbol{x}$, $\boldsymbol{y}$, z in the frame is a unit vector and, as an axis of rotation, each of these vectors is termed a *principal axis* of rotation. A rotation about a principal axis is termed a *principal rotation*. These rotations are also known as *elementary rotations*. The vector space operators: $\boldsymbol{R_x}\langle\alpha\rangle,\ \boldsymbol{R_y}\langle\beta\rangle,$ and $\boldsymbol{R_z}\langle\gamma\rangle$ denote the three principal rotations through the respective angles $\alpha,\ \beta,$ and γ.

As a consequence of Euler's rotation theorem (see 6.4.2.1), the composition of any sequence of principal rotations $\boldsymbol{R_x}\langle\alpha\rangle$, $\boldsymbol{R_y}\langle\beta\rangle$, and $\boldsymbol{R_z}\langle\gamma\rangle$ is equivalent to a single rotation $\boldsymbol{R_n}\langle\theta\rangle$. As shown in 6.4.2.4, rotation operations are not commutative, therefore the order in which the principal rotation operations are applied is important. Euler angle conventions for such principal rotation sequences are specified in 6.6.4.

In many applications, the sequence of principal rotations of an object is based on the axes of a frame that is attached to that object. The natural interpretation of such rotation sequences corresponds to the body-fixed convention given in 6.4.2.4. However, the Euler angle conventions for principal rotation sequences use the space-fixed equivalent of body-fixed convention, defined in Equation 6.4, as it is mathematically simpler.

Each of the principal rotations is defined by a 3x3 rotation matrix. The matrix representations of the principal rotations are given in Table 6.1.

Table 6.1 — Principal rotation matrices

Name	Rotation Operator	Rotation Matrix
x-axis principal rotation	$\boldsymbol{R}_x\langle\alpha\rangle$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$
y-axis principal rotation	$\boldsymbol{R}_y\langle\beta\rangle$	$\begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$
z-axis principal rotation	$\boldsymbol{R}_z\langle\gamma\rangle$	$\begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6.4.3.5 Equivalence of rotation and change of basis operators

As stated in 6.4.3.3, the rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ can be conceptually considered to have rotated a frame $\boldsymbol{F}$ away from another frame $\boldsymbol{E}$ with the same origin, and so can also be denoted by $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$. The rotation operator rotates all the basis vectors of frame $\boldsymbol{F}$ away from frame $\boldsymbol{E}$ by an angle θ about an axis $\boldsymbol{n}$. This rotation operator performs the same computation as the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ that expresses position-vectors given in terms of frame $\boldsymbol{F}$ in terms of frame $\boldsymbol{E}$. The equivalence of $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ and $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ is shown in A.12.

The semantic difference between the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ and the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ is in the interpretation of the output of the two operators. $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ conceptually rotates each of the basis vectors of frame $\boldsymbol{E}$ about axis $\boldsymbol{n}$ though angle θ to yield each of the corresponding basis vector for a new frame $\boldsymbol{F}$ (in terms of frame $\boldsymbol{E}$). The change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ re-expresses each of the basis vectors of the frame $\boldsymbol{F}$ in terms of frame $\boldsymbol{E}$.

The rotation matrix that corresponds to the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ and the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ are equivalent (see 6.2.5).

6.4.4 Rotation and Orientation

The angular relationship between two frame establishes the orientation of one frame with respect to the other. The origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ (see 6.4.3.5) that rotates the object-frame $\boldsymbol{F}$ away from alignment with the reference-frame $\boldsymbol{E}$ expresses the orientation of the object-frame $\boldsymbol{F}$ with respect to the reference-frame $\boldsymbol{E}$. Since the angular relationship is bidirectional, the inverse rotation operator $\boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}$ expresses the angular relationship between the two frames in the opposite direction, representing the orientation of the reference-frame $\boldsymbol{E}$ with respect to the object-frame $\boldsymbol{F}$.

The rotation matrix that represents the rotation operator $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ and the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ both represent the orientation of object-frame $\boldsymbol{F}$ with respect to reference-frame $\boldsymbol{E}$.

6.5 Operator summary

The operators for rotation and change of basis are closely related and can be used to perform the same functions. They commonly perform dual roles in many application domains, and can be easily confused with respect to the use of signs, inverses, rotation order, and other conventions. Both can be used to express orientation relationships between frames.

The various rotation and change of basis operators are summarized in Table 6.2. For each operator, its symbol is given, along with a brief description of its meaning, any equivalent operators, and any inverse operators.

Table 6.2 — Change of Basis, Orientation, and Rotation Operators

Operator Symbol	Meaning	Equivalent(s)	Inverse(s)
$\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$	Rotates a vector about an origin-fixed directed axis $\boldsymbol{n}$ through an angle θ	$\boldsymbol{R}_{-\boldsymbol{n}}\langle-\theta\rangle$	$\boldsymbol{R}_{\boldsymbol{n}}^{-1}\langle\theta\rangle$ $\boldsymbol{R}_{\boldsymbol{n}}\langle-\theta\rangle$ $\boldsymbol{R}_{-\boldsymbol{n}}\langle\theta\rangle$
$\boldsymbol{R}_x\langle\alpha\rangle$ $\boldsymbol{R}_y\langle\beta\rangle$ $\boldsymbol{R}_z\langle\gamma\rangle$	Rotates about the principal axes $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of a given orthonormal frame $\boldsymbol{E}$ by respective angles α, β, and γ – principal rotations	$\boldsymbol{R}_{-x}\langle-\alpha\rangle$ $\boldsymbol{R}_{-y}\langle-\beta\rangle$ $\boldsymbol{R}_{-z}\langle-\gamma\rangle$	$\boldsymbol{R}_x\langle-\alpha\rangle$ $\boldsymbol{R}_y\langle-\beta\rangle$ $\boldsymbol{R}_z\langle-\gamma\rangle$
$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]_{\boldsymbol{E}}$	Matrix form of the rotation operation that rotates vectors in frame $\boldsymbol{E}$ about an origin-fixed axis $\boldsymbol{n}$ through an angle θ	$[\boldsymbol{R}]_{\boldsymbol{E}}$	$[\boldsymbol{R}_{\boldsymbol{n}}\langle-\theta\rangle]_{\boldsymbol{E}}$
$\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$	Given common-origin frames $\boldsymbol{E}$ and $\boldsymbol{F}$ that are fully aligned, conceptually rotates frame $\boldsymbol{F}$ away from frame $\boldsymbol{E}$ (by some angle θ about some origin-fixed axis $\boldsymbol{n}$). This operator is commonly shown in matrix form.	$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]_{\boldsymbol{E}}$ $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$	$\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}^{-1} = \boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}$
$\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$	Given two common-origin frames $\boldsymbol{E}$ and $\boldsymbol{F}$ with a given rotational relationship, takes a position vector expressed in terms of $\boldsymbol{F}$ and re-expresses it in terms of $\boldsymbol{E}$. This operator is commonly shown in matrix form.	$\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]_{\boldsymbol{E}}$	$\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}^{-1} = \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$
$\boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}$	Given common-origin frames $\boldsymbol{E}$ and $\boldsymbol{F}$ that are fully aligned, conceptually rotates frame $\boldsymbol{E}$ away from frame $\boldsymbol{F}$ (by some angle θ about some origin-fixed axis $\boldsymbol{n}$). This operator is commonly shown in matrix form.	$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]_{\boldsymbol{F}}$ $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$	$\boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}^{-1} = \boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$
$\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$	Given two common-origin frames $\boldsymbol{E}$ and $\boldsymbol{F}$ with a given rotational relationship, takes a position vector expressed in terms of $\boldsymbol{E}$ and re-expresses it in terms of $\boldsymbol{F}$. This operator is commonly shown in matrix form.	$\boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}$ $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]_{\boldsymbol{F}}$	$\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}^{-1} = \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$

Given two common-origin orthonormal frames $\boldsymbol{E}$ and $\boldsymbol{F}$, $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ denotes the change of basis operation that takes a position vector expressed in terms of $\boldsymbol{F}$ and re-expresses it in terms of $\boldsymbol{E}$ (see 6.2.2).

The inverse change of basis operation that takes a position-vector expressed in terms of $\boldsymbol{E}$ and re-expresses it in terms of $\boldsymbol{F}$ is denoted by $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ (see 6.2.2). By definition, $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}} = \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}^{-1}$.

$\boldsymbol{R_n}\langle\theta\rangle$ denotes an origin-fixed rotation operation about a directed axis $\boldsymbol{n}$ through an angle θ, with positive direction of rotation defined by the right-hand rule (see 6.4.2.1).

The inverse rotation operation is accomplished by either inverting the rotation action, reversing the sign of the rotation angle, or reversing the direction of the rotation axis. These are denoted by the respective rotation operators: $\boldsymbol{R_n^{-1}}\langle\theta\rangle$, $\boldsymbol{R_n}\langle-\theta\rangle$, $\boldsymbol{R_{-n}}\langle\theta\rangle$, where the equality $\boldsymbol{R_n^{-1}}\langle\theta\rangle = \boldsymbol{R_{-n}}\langle\theta\rangle = \boldsymbol{R_n}\langle-\theta\rangle$ holds (see 6.4.2.3).

The rotation operations by an angle θ about the principal axes $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of an orthonormal frame $\boldsymbol{E}$ are denoted as the respective operators $\boldsymbol{R_x}\langle\theta\rangle$, $\boldsymbol{R_y}\langle\theta\rangle$, and $\boldsymbol{R_z}\langle\theta\rangle$ (see 6.4.3.4).

The composition $\boldsymbol{R_x}\langle\alpha\rangle \circ \boldsymbol{R_y}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$ denotes three consecutive principal rotations as the sequence of first rotating about the axis $\boldsymbol{z}$ by angle γ, then about the axis $\boldsymbol{y}$ by angle β, and finally about the axis $\boldsymbol{x}$ by angle α (see 6.4.2.4).

Given an orthonormal frame $\boldsymbol{E}$, the composition of any sequence of its principal rotations is equivalent to a single rotation about an origin-fixed axis $\boldsymbol{n}$ through an angle θ, denoted as $\boldsymbol{R_n}\langle\theta\rangle$. Expressed in matrix form, the same rotation operation is denoted as $[\boldsymbol{R_n}\langle\theta\rangle]_{\boldsymbol{E}}$. In some cases, for brevity, $[\boldsymbol{R}]_{\boldsymbol{E}}$ may be used as a substitute for $[\boldsymbol{R_n}\langle\theta\rangle]_{\boldsymbol{E}}$ (see 6.4.3.2).

Given two common-origin and fully aligned orthonormal frames $\boldsymbol{E}$ and $\boldsymbol{F}$, the rotation operation that conceptually rotates frame $\boldsymbol{F}$ away from frame $\boldsymbol{E}$ is denoted by $\boldsymbol{R_{E\to F}}$. In practice, the rotation operator rotates, one by one, each of the basis vectors of frame $\boldsymbol{F}$ away from frame $\boldsymbol{E}$ by an angle θ about an axis $\boldsymbol{n}$. The shorthand notation $\boldsymbol{R_{E\to F}}$ denotes the same equivalent operation as $[\boldsymbol{R_n}\langle\theta\rangle]_{\boldsymbol{E}}$ operating on any vector in $\boldsymbol{E}$ (see 6.4.3.3).

Stated differently, given two common-origin frames $\boldsymbol{E}$ and $\boldsymbol{F}$ with a given rotational relationship between them, the operation $\boldsymbol{R_{E\to F}}$ rotates all basis vectors of frame $\boldsymbol{E}$ by an angle θ about an axis $\boldsymbol{n}$ that will align them with their respective basis vectors of frame $\boldsymbol{F}$.

In matrix form, the rotation operator $\boldsymbol{R_{E\to F}}$ performs the same computation as the change of basis operator $\boldsymbol{\Omega_{E\leftarrow F}}$ (see 6.4.3.5).

$$\boldsymbol{R_{E\to F}} = [\boldsymbol{R_n}\langle\theta\rangle]_{\boldsymbol{E}} = \boldsymbol{\Omega_{E\leftarrow F}}$$

The semantic difference between the rotation operator $\boldsymbol{R_{E\to F}}$ and the change of basis operator $\boldsymbol{\Omega_{E\leftarrow F}}$ is in the interpretation of the output of the operation. $\boldsymbol{R_{E\to F}}$ yields the set of basis vectors for a new frame $\boldsymbol{F}$ by rotating the basis vectors of frame $\boldsymbol{E}$ about an origin-fixed axis $\boldsymbol{n}$ though angle θ. The change of basis operator $\boldsymbol{\Omega_{E\leftarrow F}}$ yields the unchanged basis vectors of frame $\boldsymbol{F}$, but expressed in terms of frame $\boldsymbol{E}$.

The rotation matrix that corresponds to the rotation operator $\boldsymbol{R_{E\to F}}$ and the direction cosine matrix that corresponds to the change of basis operator $\boldsymbol{\Omega_{E\leftarrow F}}$ are equivalent (see 6.2.5).

The orientation of frame $\boldsymbol{F}$ with respect to frame $\boldsymbol{E}$ is denoted by any of the operations $\boldsymbol{R_{E\to F}}$ or $\boldsymbol{\Omega_{E\leftarrow F}}$. Conversely, the orientation of frame $\boldsymbol{E}$ with respect to frame $\boldsymbol{F}$ is denoted by any of the operations $\boldsymbol{R_{F\to E}}$ or $\boldsymbol{\Omega_{F\leftarrow E}}$ (see 6.2.2, 6.2.3).

6.6 Representing rotations

6.6.1 Introduction

There are various ways of representing rotations. Axis-angle (see 6.6.2), matrix (see 6.6.3), Euler angle (see 6.6.4), and quaternion (see 6.6.5) representations are defined in this sub-clause. These rotation representations are also used to represent the orientation of one frame with respect to another, conceptually rotating the basis vectors of the first frame away from the second.

These representations differ in data storage size and computational complexity for rotational operations. A consequence of Euler's rotation theorem (see 6.4.3.1) is that any rotation operation on 3D Euclidean space has three degrees of freedom and may be specified by three scalar numbers. This is the minimum data storage size for a rotation and that minimum is achieved with Euler angle representations.

Other less compact specifications using four or more scalar parameters together with constraint rules are commonly used because they are more amenable to some computations and application domains, such as composing or interpolating rotations, and/or because the parameters that a particular representation uses can be measured or modelled directly. The matrix representation and the quaternion representation are in common use because the rotation of a point and the composition of rotations are directly computable as matrix or quaternion multiplications. Computing the composition of rotations in the axis-angle representation or in an Euler angle convention usually require conversion to and from matrix or quaternion forms. All rotation representations defined in the remainder of this clause tacitly require an orthonormal frame for the coordinate representation of position-vectors.

The various representation methods in prevalent use present different trade-offs with respect to storage size, computational complexity, speed, and error control. Consequently, the most appropriate representation is dependent on the requirements and computational environment of a user application. For this reason, different representations are in use and interoperability becomes an issue. This issue is compounded by the non-standard meaning of terms in prevalent use. To support interoperability and SRM operations, this International Standard defines these terms and identifies several representation methods as well as algorithms for key operations on and inter-conversions between the representation methods.

6.6.2 Axis-angle

The *axis-angle* representation $(n_1, n_2, n_3,\ \theta)$, for a given orthonormal frame, is a representation of an origin-fixed rotation $\boldsymbol{R_n}\langle\theta\rangle$ consisting of the frame coordinates of an axis of rotation unit vector $\boldsymbol{n} = [n_1,\ n_2,\ n_3]^{\mathsf{T}}$ and a rotation angle θ in radians. The axis-angle representation is not unique. In particular, the axis-angle pairs $(n_1, n_2, n_3,\ \theta)$ and $(-n_1, -n_2, -n_3,\ -\theta)$ represent the same rotation. If $\theta = 0$, $\boldsymbol{n}$ may be any unit vector or the zero vector as no rotation is indicated.

The operation of an axis-angle rotation $(n_1, n_2, n_3,\ \theta)$ on 3D Euclidean space is given by Rodrigues' rotation formula (see Equation 6.2). There is no direct computational formulation of the composition of two axis-angle rotations in axis-angle form.

NOTE 1 The axis-angle representation uses four scalar parameters $n_1,\ n_2,\ n_3$ and θ, but the unit vector constraint $\|\boldsymbol{n}\| = 1$ reduces the degrees of freedom to three.

NOTE 2 A three parameter version in the form $(a_1, a_2, a_3) = (\theta n_1, \theta n_2, \theta n_3) = \theta\boldsymbol{n}$ is also in use. In this form, θ is non-negative and is computed as $\theta = \|(a_1, a_2, a_3)\|$ and $\boldsymbol{n} = \frac{1}{\theta}(a_1, a_2, a_3)$ when $\theta \neq 0$.

6.6.3 Matrix

In $\mathbb{R}^3$, a 3x3 matrix $\boldsymbol{M}$ is a *rotation matrix* if it satisfies these properties:

$$\det(\boldsymbol{M}) = 1$$

$$\boldsymbol{M}^{-1} = \boldsymbol{M}^{\top}$$

Matrices satisfying these properties form an algebraic group with respect to matrix multiplication. This group is known as the *special orthogonal group* of degree 3, SO(3). In particular, the product of any two rotation matrices is itself a rotation matrix.

In an orthonormal frame, the operation of left matrix multiplication by a rotation matrix $\boldsymbol{M}$ corresponds to an origin-fixed rotation (see 6.4.2.1).

The direction cosine matrix that arises from a change of basis operation (see **Error! Reference source not found.**) is a rotation matrix.

NOTE 1 Matrix multiplication is generally not commutative.

NOTE 2 A 3x3 rotation matrix has nine parameters, but the constraints on the determinant and the transpose reduce the degrees of freedom to three.

6.6.4 Euler angles

6.6.4.1 Principal rotations

Euler angles are defined in terms of principal rotations (see 6.4.3.4). The Euler angle representation of a rotation is important, in part, because most inertial systems produce Euler angles as output. The vector space operators: $\boldsymbol{R_x}\langle\alpha\rangle$, $\boldsymbol{R_y}\langle\beta\rangle$, and $\boldsymbol{R_z}\langle\gamma\rangle$ denote the three principal rotations through the respective angles α, β, and γ. The axis-angle representation of the principal rotations in the given frame are, respectively, $(1, 0, 0, \alpha)$, $(0, 1, 0, \beta)$, and $(0, 0, 1, \gamma)$. Euler angles are often used to determine orientation in control mechanisms such as robotic arms and motion platforms.

6.6.4.2 Euler angle conventions

Euler angles are a specification of a rotation obtained by the composition of three consecutive principal rotations in the body-fixed convention (see 6.4.2.4). Allowing for repeated axes, there are twelve distinct ways to select a sequence of three principal axes and apply the principal rotations (24 if left-handed axes are considered)[2]. Each such ordered selection of axes is termed an *Euler angle convention*.

There is little agreement among authors on names or notations for these conventions. There are numerous Euler angle conventions in use and many are named inconsistently. The ***z–x–z*** convention defined in 6.6.4.3 is also known as the ***3-1-3*** convention or the ***x***-convention. Replacing ***x*** with ***y*** gives the so-called ***y***-convention (***z–y–z*** or ***3-2-3***). Quantum physics treatments prefer the ***y***-convention, but ***x–y–x*** (or ***1-2-1***) is also called the ***y***-convention by some authors. The ***x–y–z*** (or ***1-2-3***) convention is defined in 6.6.4.4. Some applications use left-handed coordinate systems. All orthonormal frames in this International Standard are right-handed.

[2] There cannot be two consecutive rotations on the same axis as they would combine to a single rotation. Thus, among right-handed axis systems, there are 3 choices for the first rotation axis, 2 choices each for the second and third rotation axes to avoid repeating the preceding axis choice (3x2x2=12).

This International Standard adopts the following notation: A sequence of symbols in the form A-B-C, where each symbol A, B, C is one of the principal axis letters x, y, or z, shall denote the body-fixed sequence (see Equation 6.4) of principal axis rotations beginning with the principal rotation about axis A, followed by rotated axis B, followed by doubly-rotated axis C. With this notation, the twelve distinct sequences of principal axis rotations are:

z-x-z z-y-z y-z-y y-x-y x-z-x x-y-x (Proper Euler angles)

x-y-z z-y-x y-x-z z-x-y y-z-x x-z-y (Tait-Bryan angles)

The Euler angle conventions that repeat the first principal axis for the third axis are also known as *proper Euler angles*. These sequences help avoid gimbal lock issues (see 6.6.4.5). Euler angle conventions that use all three principal axes are sometimes referred to as *Tait-Bryan angles*.

Clause 6.6.4.3 deals with the Euler z-x-z convention as representative of the Proper Euler angle conventions which use two of the three principal axes. Clause 6.6.4.4 deals with the Tait-Bryan conventions x-y-z and z-y-x, which are in wide use.

The sequence of principal rotations written as A-B-C, with respective Euler angles (α, β, γ), denotes the body-fixed convention sequence in which these rotations are applied in the order A, then B', then C', i.e., $\boldsymbol{R}_{C''}\langle\gamma\rangle \circ \boldsymbol{R}_{B'}\langle\beta\rangle \circ \boldsymbol{R}_{A}\langle\alpha\rangle$. In the space-fixed equivalent of body-fixed convention (defined in 6.4.2.4), these rotations are applied in the order C, then B, then A, i.e., $\boldsymbol{R}_{A}\langle\alpha\rangle \circ \boldsymbol{R}_{B}\langle\beta\rangle \circ \boldsymbol{R}_{C}\langle\gamma\rangle$. In the remainder of this International Standard, unless indicated otherwise, the space-fixed equivalent of body-fixed convention is used to simplify computations.

EXAMPLE The Euler sequence (ψ, θ, φ) in the Euler z-y-x convention is the composition operator $\boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$ (in the space-fixed equivalent of body-fixed convention).

In an Euler angle convention, the three angles representing a rotation are not necessarily unique modulo 2π. The conditions that result in non-unique angle 3-tuples are given in Table 6.5 for the Euler z-x-z convention and in Table 6.8 for the Euler x-y-z and z-y-x conventions (see also 6.6.4.5).

There are no direct computational formulations for the operation of an Euler angle rotation on 3D Euclidean space or for representing an Euler angle rotation sequence as a single axis-angle rotation. For these computations, the principal rotation sequence is commonly realized as a product of matrices or quaternions (see also 6.6.5.1).

6.6.4.3 The Euler z-x-z convention

In the Euler z-x-z convention, the three angles specify the principal rotations in the body-fixed composition of the z-axis principal rotation, followed by (the rotated) x'-axis principal rotation, followed by the (twice rotated) z''-axis principal rotation. The initial xy-plane and the final rotated $x''y''$-plane intersect in a line. This line is termed the *line of nodes* for this convention.

The three Euler z-x-z convention angles are defined as follows:

α is the angle between the line of nodes and the x''-axis,

β is the angle between the z-axis and the z''-axis, and

γ is the angle between the x-axis and the line of nodes.

In some contexts, the angles $\alpha,\ \beta,\ \gamma$ are known, respectively, as the *spin* angle, the *nutation* angle, and the *precession* angle.

In the case that the initial $\boldsymbol{xy}$-plane lies in the final rotated $\boldsymbol{x''y''}$-plane, $\beta = 0$ or $\beta = \pi$ (see 6.6.4.5).

The Euler z-x-z convention is common in robotics because it maps to the sequence of rotations provided by many six-axis robotic manipulators. This convention facilitates the achievement of the desired position and orientation of the manipulator's end-effector.

The sequence of body-fixed rotations is illustrated in Figure 6.11. The resulting composite rotation operation is $\boldsymbol{R}_{\boldsymbol{z}''}\langle\gamma\rangle \circ \boldsymbol{R}_{\boldsymbol{x}'}\langle\beta\rangle \circ \boldsymbol{R}_{\boldsymbol{z}}\langle\alpha\rangle$ in the body-fixed convention, or $\boldsymbol{R}_{\boldsymbol{z}}\langle\alpha\rangle \circ \boldsymbol{R}_{\boldsymbol{x}}\langle\beta\rangle \circ \boldsymbol{R}_{\boldsymbol{z}}\langle\gamma\rangle$ in the space-fixed equivalent of body-fixed convention.

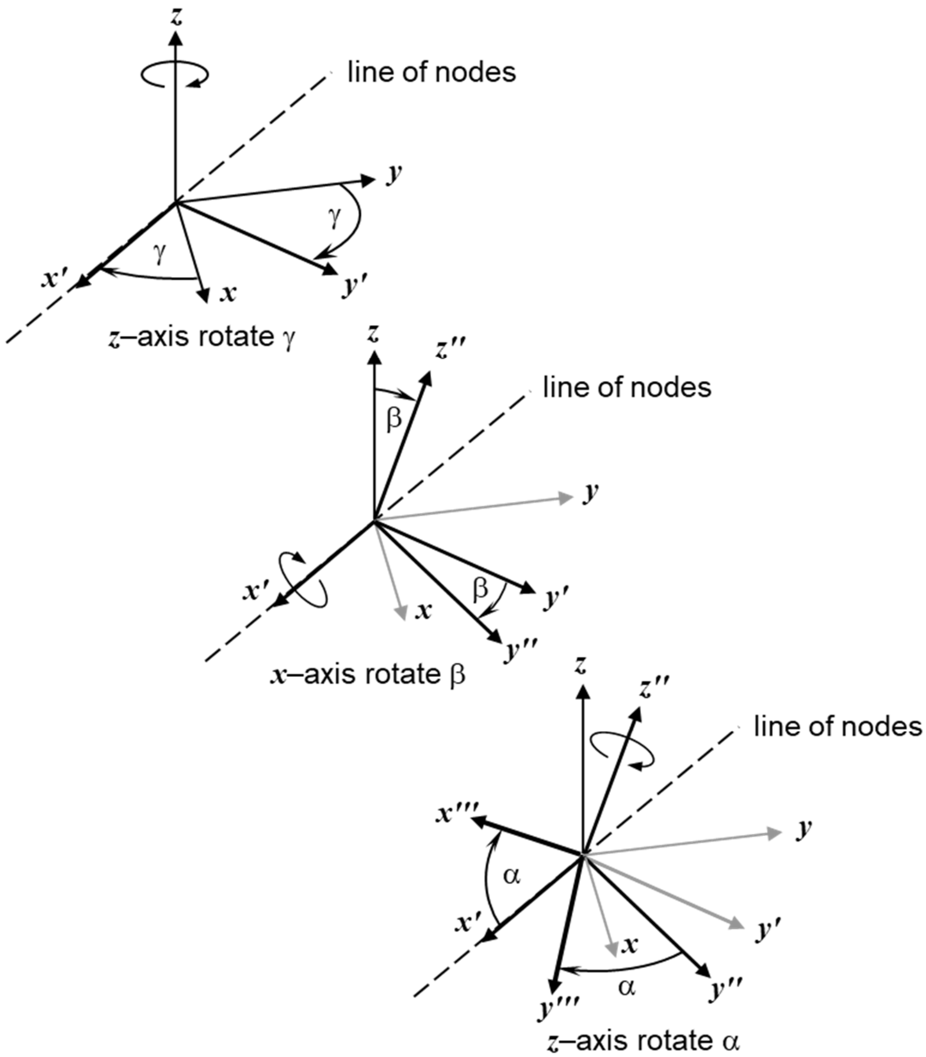

Figure 6.11 — Euler z-x-z convention rotation sequence

6.6.4.4 Tait-Bryan angles

The angles in the Euler x-y-z and z-y-x conventions are variously termed Tait-Bryan angles, *Cardano angles*, or *nautical angles*. The various names given to these angles include:

φ roll or bank or tilt,

θ pitch or elevation, and

ψ yaw or heading or azimuth.

Tait-Bryan angles are commonly used in control systems, navigation, and vehicle simulations (see Figure 6.12).

Case x-y-z:

In the Euler x-y-z convention the line of nodes is the intersection of the xy-plane and the final rotated $y''z''$-plane. The Euler x-y-z convention angles are defined as follows:

φ is the angle between the line of nodes and the y''-axis,

θ is the angle between x''-axis and the xy-plane, (equivalently, the z-axis and the $y''z''$-plane), and

ψ is the angle between the y-axis and the line of nodes.

The resulting composite rotation operation is $\boldsymbol{R}_{z''}\langle\psi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$ in the body-fixed convention, or $\boldsymbol{R}_{x}\langle\varphi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$ in the space-fixed equivalent of body-fixed convention.

Case z-y-x:

In the Euler z-y-x convention the line of nodes is the intersection of the yz-plane and the final rotated $x''y''$-plane. The Euler z-y-x convention angles are defined as follows:

φ is the angle between the line of nodes and the y-axis,

θ is the angle between x-axis and the $x''y''$-plane, (equivalently, the z''-axis and the yz-plane), and

ψ is the angle between the y''-axis and the line of nodes.

The resulting composite rotation operation is $\boldsymbol{R}_{x''}\langle\varphi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$ in the body-fixed convention, or $\boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$ in the space-fixed equivalent of body-fixed convention.

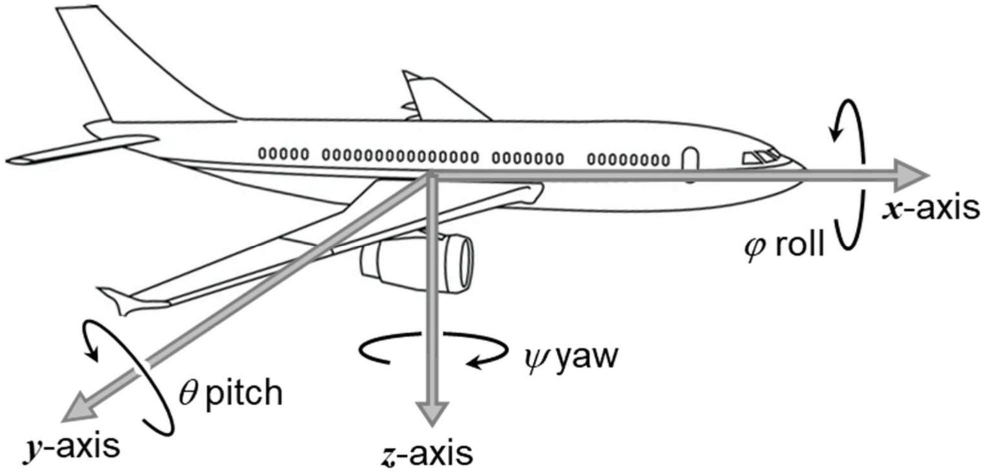

Figure 6.12 — Tait-Bryan angles

6.6.4.5 Gimbal lock

The term *gimbal lock* refers to a situation when an n-nested set of gimbals are at critical angles, the rotational degrees of freedom are reduced from n to n-1. Each gyroscope mounting scheme corresponds to an Euler angle convention. In any such mounting scheme, there exist critical angles for the middle gimbal that reduce the rotational degrees of freedom from three to two. In those critical configurations, the gimbals lie in a single plane and rotation within that plane is figuratively "locked out" by the gimbal mechanism. This loss of a degree of freedom is termed "gimbal lock".

Case z-x-z:

In the case of the Euler z-x-z convention, it is assumed that the xy-plane and $x''y''$-plane intersect in a line (the line on nodes). That assumption is met when $\beta \neq 0$ (modulo 2π) and $\beta \neq \pi$. If $\beta = 0$, $\boldsymbol{R}_x\langle 0\rangle$ is the identity operator and has no effect. If $\beta = \pi$, $\boldsymbol{R}_x\langle\pi\rangle$ reverses the direction of the preceding z-axis rotation so that $\boldsymbol{R}_x\langle\pi\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle-\gamma\rangle \circ \boldsymbol{R}_x\langle\pi\rangle$. In either case, the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned}
&\beta = 0: \quad \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle 0\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle\alpha+\gamma\rangle \\
&\beta = \pi: \quad \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\pi\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_z\langle-\gamma\rangle \circ \boldsymbol{R}_x\langle\pi\rangle = \boldsymbol{R}_z\langle\alpha-\gamma\rangle \circ \boldsymbol{R}_x\langle\pi\rangle.
\end{aligned} \tag{6.7}$$

EXAMPLE 1 This situation is illustrated by a spinning top. The top spins on its spin-axis and precesses about the precession-axis. The angle between the spin- and precession-axes is the nutation angle. When the spin-axis is perfectly vertical (either upright or upside down), the nutation angle is 0 or π and the spin- and precession-axes become indistinguishable from each other as indicated in Equation 6.7.

Case x-y-z:

In the case of the Euler x-y-z convention (Tait-Bryan) it is assumed that the xy-plane and $y''z''$-plane intersect in a line (the line of nodes). That assumption is met when $\theta \neq \pm\pi/2$ modulo 2π. When $\theta = \pm\pi/2$ and the x''-axis becomes parallel to the z-axis and the consecutive rotations collapse down to a single principal rotation:

$$\theta = +\pi/2: \quad \boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \circ \boldsymbol{R}_z\langle\psi\rangle = \boldsymbol{R}_x\langle\varphi+\psi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \tag{6.8}$$

$$\theta = -\pi/2: \quad \boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle \circ \boldsymbol{R}_z\langle\psi\rangle = \boldsymbol{R}_x\langle\varphi - \psi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle.$$

Case $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$:

The case of the Euler $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention (Tait-Bryan) has a similar result:

$$\begin{aligned} \theta &= +\pi/2: \boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle = \boldsymbol{R}_z\langle\psi + \varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \\ \theta &= -\pi/2: \boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle = \boldsymbol{R}_z\langle\psi - \varphi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle \end{aligned} \qquad (6.9)$$

EXAMPLE 2 This situation is illustrated by an aircraft as in Figure 6.12. When the aircraft either climbs vertically, or dives vertically, roll-rotation cannot be distinguished from (plus or minus) yaw-rotation. This occurs at critical pitch angles of $\theta = \pm\pi/2$ as indicated in Equations 6.8 and 6.9.

6.6.5 Quaternions

6.6.5.1 Quaternion notations and conventions

The quaternion system is a 4-dimensional vector space together with a vector multiplication operation that forms a non-commutative associative algebra. In analogy to complex numbers that are written as $a + \boldsymbol{i}b$, $\boldsymbol{i}^2 = -1$, quaternion axes $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are defined with the following relationships: $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. There are several notational conventions in use including the three termed in this International Standard as the *Hamilton form*, the *4-tuple form*, and the *scalar vector form*. In these notation forms, a quaternion $\boldsymbol{q}$ is denoted as follows:

$\boldsymbol{q} = e_0 + e_1\boldsymbol{i} + e_2\boldsymbol{j} + e_3\boldsymbol{k}$	Hamilton form
$\boldsymbol{q} = (e_0, e_1, e_2, e_3)$	4-tuple form
$\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = [e_1,\ e_2,\ e_3]^{\mathrm{T}}$	scalar vector form

where e_0, e_1, e_2, e_3 are scalar values.

The e_0 value is termed the *real* (or "scalar") part of $\boldsymbol{q}$ and (e_1, e_2, e_3) is termed the *imaginary* (or "vector") part of $\boldsymbol{q}$. The remainder of this clause uses the scalar vector form.

NOTE 1 In the literature, the component order of the scalar vector form is sometimes reversed: $\boldsymbol{q} = (\boldsymbol{e}, e_0)$.

NOTE 2 A unit quaternion (see below) in 4-tuple form is also termed the *Euler parameters* (or the *Euler-Rodrigues parameters*) of a rotation. In the literature, the real part of the 4-tuple form is sometimes placed last: $\boldsymbol{q} = (e_1, e_2, e_3, e_4)$ where $e_4 = e_0$.

The quaternion representation of rotation is often used in flight-based systems since it avoids gimbal lock. The quaternion representation of rotation also facilitates the computation of interpolated rotation between two rotations.

The principal rotations (see 6.6.4.1) correspond to the following quaternions:

$$\boldsymbol{R}_x\langle\alpha\rangle \leftrightarrow (\cos(\alpha/2),\ \sin(\alpha/2)\,\boldsymbol{x})$$

$$\boldsymbol{R}_y\langle\beta\rangle \leftrightarrow (\cos(\beta/2),\ \sin(\beta/2)\,\boldsymbol{y})$$

$$\boldsymbol{R}_z\langle\gamma\rangle \leftrightarrow (\cos(\gamma/2),\ \sin(\gamma/2)\,\boldsymbol{z})$$

For each Euler angle convention, multiply the corresponding quaternions in body-fixed composition order. Terms in the resulting product may be simplified using the orthonormal property of the vector set $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and various trigonometric identities.

6.6.5.2 Quaternion algebra

Quaternion multiplication and other operations are defined in A.10 for all three notational forms. Given two quaternions $\boldsymbol{q} = (e_0, \boldsymbol{e})$ and $\boldsymbol{p} = (d_0, \boldsymbol{d})$, A.10 defines:

the product: $\boldsymbol{pq} = \big((d_0 e_0 - \boldsymbol{d} \bullet \boldsymbol{e}),\ (e_0 \boldsymbol{d} + d_0 \boldsymbol{e} + \boldsymbol{d} \times \boldsymbol{e})\big)$,

the conjugate: $\boldsymbol{q}^* = (e_0, -\boldsymbol{e})$, and

the modulus: $|\boldsymbol{q}| = \sqrt{\boldsymbol{qq}^*} = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2}$,

where $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = (e_0^2 + e_1^2 + e_2^2 + e_3^2,\ \ \boldsymbol{0})$.

A quaternion $\boldsymbol{q}$ is a *unit quaternion* if $|\boldsymbol{q}| = 1$. In that case $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = (1,\ \boldsymbol{0})$ which is the multiplicative identity so that, for a unit quaternion, its conjugate is its multiplicative inverse $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$. Any unit quaternion may be expressed in the form:

$$\boldsymbol{q} = (\cos(\theta/2)\,,\ \sin(\theta/2)\,\boldsymbol{n}) \tag{6.10}$$

where:

$\boldsymbol{n} = \frac{1}{\|\boldsymbol{e}\|}\boldsymbol{e}$ is a unit vector in 3D space.

$\theta = 2 \cdot \operatorname{arctan2}\left(\sqrt{e_1^2 + e_2^2 + e_3^2},\ e_0\right)$.

NOTE The two argument arctangent function arctan2() is defined in A.8.1

6.6.5.3 Quaternion operators on 3D Euclidean space

Each quaternion $\boldsymbol{q}$ corresponds to a transformation of 3D Euclidean space as follows. If $\boldsymbol{r}$ is a position-vector in 3D Euclidean space, the corresponding quaternion is formed by using 0 for the real part and $\boldsymbol{r}$ for the imaginary part $(0, \boldsymbol{r})$. A unit quaternion $\boldsymbol{q}$ operates on $(0, \boldsymbol{r})$ by left multiplying with $\boldsymbol{q}$ and right multiplying with its conjugate $\boldsymbol{q}^*$. The real part of the product, $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^* = (\boldsymbol{r'}_0, \boldsymbol{r'})$, is 0. Thus, $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^* = (0, \boldsymbol{r'})$ is purely imaginary and the quaternion $\boldsymbol{q}$ associates $\boldsymbol{r'}$ with $\boldsymbol{r}$. Symbolically the operation on $\boldsymbol{r}$ is:

$$\boldsymbol{r} \mapsto \boldsymbol{r'} = \text{imaginary part}\{\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*\}.$$

This is equivalent to:

$$\boldsymbol{r'} = (e_0^2 - \boldsymbol{e} \bullet \boldsymbol{e})\boldsymbol{r} + 2(\boldsymbol{e} \bullet \boldsymbol{r})\boldsymbol{e} + 2e_0 \boldsymbol{e} \times \boldsymbol{r}. \tag{6.11}$$

$-\boldsymbol{q} = (-e_0, -\boldsymbol{e})$ produces the same $\boldsymbol{r}'$ so that $\boldsymbol{q}$ and $-\boldsymbol{q}$ produce equivalent rotations.

If $\boldsymbol{q} = \left(\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right)$ is a unit quaternion, Equation 6.11 reduces to the Rodrigues rotation formula for a clockwise rotation about $\boldsymbol{n}$ through angle θ: $\boldsymbol{r'} = \cos(\theta)\,\boldsymbol{r} + (1 - \cos(\theta))(\boldsymbol{n} \bullet \boldsymbol{r})\boldsymbol{n} + \sin(\theta)\,\boldsymbol{n} \times \boldsymbol{r}$.

A non-zero quaternion $\boldsymbol{p}$ and its corresponding unit quaternion $\boldsymbol{q} = \boldsymbol{p}/|\boldsymbol{p}|$ perform the same rotation $\boldsymbol{p}(0,\boldsymbol{r})\boldsymbol{p}^{-1} = \boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^{*}$. For this reason, some authors use $\boldsymbol{p}(0,\boldsymbol{r})\boldsymbol{p}^{-1}$ operations for any non-zero quaternion while others use the $\boldsymbol{q}(0,\boldsymbol{r})\boldsymbol{q}^{*}$ operator and restrict operations only to unit quaternions.

The quaternion representation of rotation facilitates the computation of the composition of rotations and the interpolation between two rotations. If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are two unit quaternions, the composite rotation on $\boldsymbol{r}$ that is obtained by first rotating with the rotation induced by $\boldsymbol{q}_1$ and then rotating the result with the rotation induced by $\boldsymbol{q}_2$ is the same as the single rotation induced by the product $\boldsymbol{q}_2\boldsymbol{q}_1$ since $\boldsymbol{q}_2\{\boldsymbol{q}_1(0,\boldsymbol{r})\boldsymbol{q}_1^{*}\}\boldsymbol{q}_2^{*} = \boldsymbol{q}_2\boldsymbol{q}_1(0,\boldsymbol{r})\boldsymbol{q}_1^{*}\boldsymbol{q}_2^{*} = \{\boldsymbol{q}_2\boldsymbol{q}_1\}(0,\boldsymbol{r})\{\boldsymbol{q}_2\boldsymbol{q}_1\}^{*}$.

6.6.6 Representation summary

Important attributes of the representations in this section are summarized in Table 6.3.

Table 6.3 — Summary of representation attributes

Representation type	Data compo-nents	Data constraints	Ambiguities (modulo 2π)	Composition	Inverse
Axis-angle (n_1, n_2, n_3, θ)	4	$\lVert\boldsymbol{n}\rVert = 1$ $\boldsymbol{n} = [n_1, n_2, n_3]^{\mathrm{T}}$	(n_1, n_2, n_3, θ) is equivalent to $(-n_1, -n_2, -n_3, -\theta)$. If $\theta = 0$, $\boldsymbol{n}$ is indeterminate	Convert to/from matrix or quaternion to compose	$(n_1, n_2, n_3, -\theta)$ or $(-n_1, -n_2, -n_3, \theta)$
Matrix $\boldsymbol{M}$	9	$\det(\boldsymbol{M}) = 1$ $\boldsymbol{M}^{-1} = \boldsymbol{M}^{\mathrm{T}}$	None	Matrix multiplication	$\boldsymbol{M}^{\mathrm{T}}$
Euler angles	3	None	2 or more *z-x-z* convention: see Table 6.5 Tait-Bryan *z-y-x* or *x-y-z* angles: see Table 6.8	Convert to/from matrix or quaternion to compose (see Note 2)	See Note 1
Quaternion $\boldsymbol{q}$	4	unit constraint: $\boldsymbol{q}\boldsymbol{q}^{*} = 1$	$\boldsymbol{q}$ is equivalent to $-\boldsymbol{q}$ (see Note 3)	Quaternion multiplication	$\boldsymbol{q}^{*}$ or $-\boldsymbol{q}^{*}$

NOTE 1 The inverse in the Euler *z-x-z* convention is:

$$[\boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\beta\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle]^{-1} = \boldsymbol{R}_z\langle-\gamma\rangle \circ \boldsymbol{R}_x\langle-\beta\rangle \circ \boldsymbol{R}_z\langle-\alpha\rangle.$$

The inverse in the Euler *x-y-z* and *z-y-x* conventions (Tait-Bryan angles) are

$$[\boldsymbol{R}_x\langle\phi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_z\langle\psi\rangle]^{-1} = \boldsymbol{R}_z\langle-\psi\rangle \circ \boldsymbol{R}_y\langle-\theta\rangle \circ \boldsymbol{R}_x\langle-\phi\rangle$$

$$[\boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\phi\rangle]^{-1} = \boldsymbol{R}_x\langle-\phi\rangle \circ \boldsymbol{R}_y\langle-\theta\rangle \circ \boldsymbol{R}_z\langle-\psi\rangle$$

NOTE 2 The composition of Euler angle operations may also be performed in a "direct" method that involves lengthy expressions combining forward and inverse trigonometric functions.

NOTE 3 Formulae such as Equation 6.10 require the unit quaternion constraint. Other useful relationships such as Equation 6.11 do not have that requirement. For that reason, some applications do not enforce the unit constraint. In the unconstrained case, every non-zero scalar multiple of a given quaternion is rotationally equivalent to it.

6.7 Rotation representation conversions

6.7.1 Introduction

Direct conversions between each pair of rotation representations in 6.6 are found in this sub-clause with the exception of conversions between axis-angle and Euler angle representations. These conversions are more effectively performed in two steps, using matrix or quaternion forms as intermediate representations.

6.7.2 From axis-angle to matrix

Given the axis-angle rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ parameters (n_1, n_2, n_3, θ), the corresponding rotation matrix $\boldsymbol{M}$ is given by the matrix form of Rodrigues' rotation formula (see Equation 6.6):

$$\boldsymbol{M} = \begin{bmatrix} (1-\cos\theta)n_1^2 + \cos\theta & (1-\cos\theta)n_1n_2 - n_3\sin\theta & (1-\cos\theta)n_1n_3 + n_2\sin\theta \\ (1-\cos\theta)n_2n_1 + n_3\sin\theta & (1-\cos\theta)n_2^2 + \cos\theta & (1-\cos\theta)n_2n_3 - n_1\sin\theta \\ (1-\cos\theta)n_3n_1 - n_2\sin\theta & (1-\cos\theta)n_3n_2 + n_1\sin\theta & (1-\cos\theta)n_3^2 + \cos\theta \end{bmatrix}$$

6.7.3 From matrix to axis-angle

Given a rotation matrix $\boldsymbol{M}$ with elements a_{ij}, the corresponding axis-angle rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ parameters $(n_1, n_2, n_3,\ \theta)$ are algorithmically determined as follows:

If $\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\theta = \arccos\left(\frac{(a_{11}+a_{22}+a_{33})-1}{2}\right), \quad 0 \le \theta \le \pi.$

There are three cases for the computation of $\boldsymbol{n} = (n_1, n_2, n_3)^{\mathsf{T}}$ that depend on the value of θ.

Case $\theta = 0$: There is no rotation, so $\boldsymbol{n}$ is indeterminate.

Case $0 < \theta < \pi$: Let $\boldsymbol{n} = \boldsymbol{v}/\|\boldsymbol{v}\|$, where:

$\boldsymbol{v} = \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$. In this case, $\|\boldsymbol{v}\| = 2\,|\sin(\theta)|$.

Case: $\theta = \pi$:

Find the maximum diagonal element $a_{11},\ a_{22}$, or a_{33} of $\boldsymbol{M}$.

Let $\boldsymbol{n} = \boldsymbol{v}/\|\boldsymbol{v}\|$, where:

Sub-case a_{11} is the maximum: $\boldsymbol{v} = [a_{11}+1,\ a_{12},\ a_{13}]^{\mathsf{T}}$.

Sub-case a_{22} is the maximum: $\boldsymbol{v} = [a_{21},\ a_{22}+1,\ a_{23}]^{\mathsf{T}}$.

Sub-case a_{33} is the maximum: $\boldsymbol{v} = [a_{31},\ a_{32},\ a_{33}+1]^{\mathsf{T}}$.

In all cases $(-n_1, -n_2, -n_3, -\theta)$ is also a solution.

6.7.4 From Euler angle z-x-z convention to matrix

Given the Euler angle z-x-z convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_z}\langle\alpha\rangle \circ \boldsymbol{R_x}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$, the corresponding rotation matrix $\boldsymbol{M}$ is the matrix product of the three principal rotation matrices specified in Table 6.1. The resulting matrix is given in Equation 6.12.

$$\boldsymbol{M} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\sin\gamma\cos\alpha - \cos\beta\cos\gamma\sin\alpha & \sin\beta\sin\alpha \\ \cos\beta\sin\gamma\cos\alpha + \cos\gamma\sin\alpha & \cos\beta\cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -\sin\beta\cos\alpha \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{bmatrix} \tag{6.12}$$

6.7.5 From matrix to Euler angle z-x-z convention

Given a rotation matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may be solved for the principal rotation factors $\boldsymbol{R_z}\langle\alpha\rangle \circ \boldsymbol{R_x}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$, and therefore solved for angles (α, β, γ). The solution is given in Table 6.4.

Table 6.4 — Principal rotation factors for the Euler angle z-x-z convention

Case	Principal rotation factors for $\boldsymbol{R_z}\langle\alpha\rangle \circ \boldsymbol{R_x}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$ (all angles modulo 2π, $M = [a_{ij}]$)		
$a_{33} \neq \pm 1$	$\beta = \arccos(a_{33})$ [principal value] $0 < \beta < \pi$	$\gamma = \operatorname{arctan2}(a_{31},\ a_{32})$	$\alpha = \operatorname{arctan2}(a_{13},\ -a_{23})$
	$\beta = \arccos(a_{33})$ [2π – principal value] $\pi < \beta < 2\pi$	$\gamma = \operatorname{arctan2}(-a_{31},\ -a_{32})$	$\alpha = \operatorname{arctan2}(-a_{13},\ a_{23})$
$a_{33} = -1$	$\beta = \pi$	any value of γ	$\alpha = \operatorname{arctan2}(a_{21},\ a_{11}) + \gamma$
$a_{33} = +1$	$\beta = 0$	any value of γ	$\alpha = \operatorname{arctan2}(a_{21},\ a_{11}) - \gamma$

In the case $a_{33} \neq \pm 1$, $\arccos(\,)$ is multi-valued so that there are two valid solution sets depending on the quadrants selected for arccosine values[3]. The principal value solution is the commonly used one. The two argument arctangent function $\operatorname{arctan2}(\,)$ is defined in A.8.1

The cases $a_{33} = -1$ with $\beta = \pi$ and $a_{33} = +1$ with $\beta = 0$ are gimbal lock cases (see Equation 6.7). The corresponding rotations and simplified matrices are:

Case $a_{33} = -1,\ \beta = \pi$: $\boldsymbol{R_z}\langle\alpha - \gamma\rangle \circ \boldsymbol{R_x}\langle\pi\rangle$ $M = \begin{bmatrix} \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ \sin(\alpha-\gamma) & -\cos(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \end{bmatrix}$

[3] Computer library functions such as $\mathrm{acos}()$ return the principal value only. The second solution for β may be obtained by subtracting the principal value from 2π.

Case $a_{33} = +1,\ \beta = 0$: $\boldsymbol{R_z}\langle \alpha + \gamma \rangle$ $M = \begin{bmatrix} \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & 0 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In case $a_{33} = +1$ only the difference of the other two angles can be determined by using $\alpha - \gamma = \operatorname{arctan2}(a_{21},\ a_{11})$. Therefore, all values are valid for α if $\gamma = \operatorname{arctan2}(a_{21},\ a_{11}) + \alpha$. The case $a_{31} = +1$ is similar to the previous case with the sum of the angles determined by using $\gamma = \operatorname{arctan2}(a_{21},\ a_{11}) - \alpha$.

As seen in Table 6.4, the three-angle sequence corresponding to a given rotation operator is not unique modulo 2π. Two sequences, $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ of z-x-z principal rotation factors specify the same operator if they satisfy one of the criteria specified in Table 6.5.

Table 6.5 — Equivalence of z-x-z principal rotation factor sequences

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ for the Euler z-x-z convention**		
$\beta_1 = \beta_2$	$\alpha_1 = \alpha_2,\ \gamma_1 = \gamma_2$	$[\beta_1, \beta_2 \neq 0 \text{ or } \pi]$	(in)equalities modulo 2π
$\lvert\beta_1 + \beta_2\rvert = 2\pi$	$\lvert\alpha_2 - \alpha_1\rvert = \pi,\ \ \lvert\gamma_2 - \gamma_1\rvert = \pi$	$[\beta_1, \beta_2 \neq 0 \text{ or } \pi]$	(in)equalities modulo 2π
$\beta_1 = \beta_2 = \pi$	$\alpha_1 - \gamma_1 = \alpha_2 - \gamma_2$		equality modulo 2π
$\beta_1 = \beta_2 = 0$	$\alpha_1 + \gamma_1 = \alpha_2 + \gamma_2$		equality modulo 2π

6.7.6 From Tait-Bryan angle x-y-z convention to matrix

Given the Tait-Bryan x-y-z convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_x}\langle\varphi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_z}\langle\psi\rangle$, the corresponding rotation matrix $\boldsymbol{M}$ is the matrix product of the three principal rotation matrices specified in Table 6.1. The resulting matrix is given in Equation 6.13.

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\theta & \sin\theta \\ \cos\psi\sin\theta\sin\varphi + \sin\psi\cos\varphi & -\sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & -\cos\theta\sin\varphi \\ -\cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi & \sin\psi\sin\theta\cos\varphi + \cos\psi\sin\varphi & \cos\theta\cos\varphi \end{bmatrix} \tag{6.13}$$

6.7.7 From matrix to Tait-Bryan angle x-y-z convention

Given a rotation matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may be solved for the principal rotation factors $\boldsymbol{R_x}\langle\varphi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_z}\langle\psi\rangle$, and therefore solved for angles (φ, θ, ψ). The solution is given in Table 6.6.

Table 6.6 — Principal rotation factors for the Tait-Bryan angle *x-y-z* convention

Case	Principal rotation factors for $\boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_z\langle\psi\rangle$ (all angles modulo 2π, $M = [a_{ij}]$)		
$a_{13} \neq \pm 1$	$\theta = \arcsin(a_{13})$ [principal value] $-\pi/2 < \theta < \pi/2$	$\varphi = \operatorname{arctan2}(-a_{23},\ a_{33})$	$\psi = \operatorname{arctan2}(-a_{12},\ a_{11})$
	$\theta = \arcsin(a_{13})$ [π – princ.l val.] $\pi/2 < \theta < 3\pi/2$	$\varphi = \operatorname{arctan2}(a_{23},\ -a_{33})$	$\psi = \operatorname{arctan2}(a_{12},\ -a_{11})$
$a_{13} = +1$	$\theta = \pi/2$	$\varphi = \operatorname{arctan2}(a_{21},\ -a_{31}) - \psi$	any value of ψ
$a_{13} = -1$	$\theta = -\pi/2$	$\varphi = \operatorname{arctan2}(a_{21},\ a_{31}) + \psi$	any value of ψ

In the case $a_{13} \neq \pm 1$, arcsin() is multi-valued so that there are two valid solution sets depending on the quadrant selected for arcsine values[4]. The principal value solution is the commonly used one.

The cases $a_{13} = +1$ with $\theta = \pi/2$ and $a_{13} = -1$ with $\theta = -\pi/2$ are gimbal lock cases (see Equation 6.8). The corresponding rotations and simplified matrices are:

Case $a_{13} = +1,\ \theta = +\pi/2$: $\boldsymbol{R}_x\langle\varphi + \psi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle$

$$M = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\varphi+\psi) & \cos(\varphi+\psi) & 0 \\ -\cos(\varphi+\psi) & \sin(\varphi+\psi) & 0 \end{bmatrix}$$

Case $a_{13} = -1, \theta = -\pi/2$: $\boldsymbol{R}_x\langle\varphi - \psi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle$

$$M = \begin{bmatrix} 0 & 0 & -1 \\ -\sin(\varphi-\psi) & \cos(\varphi-\psi) & 0 \\ \cos(\varphi-\psi) & \sin(\varphi-\psi) & 0 \end{bmatrix}$$

For this reason, only the sum of the other two angles is determined as $\varphi + \psi = \operatorname{arctan2}(a_{21},\ -a_{31})$. Therefore, all values are valid for ψ if we set $\varphi = \operatorname{arctan2}(a_{21},\ -a_{31}) - \psi$. The case $a_{13} = -1$ is similar to the previous case with the difference of the angles determined by $\varphi - \psi = \operatorname{arctan2}(a_{21},\ a_{31})$.

6.7.8 From Tait-Bryan angle *z-y-x* convention to matrix

Given the Tait-Bryan *z-y-x* convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle$, the corresponding rotation matrix $\boldsymbol{M}$ is the matrix product of the three principal rotation matrices specified in Table 6.1. The resulting matrix is given in Equation 6.14.

$$\boldsymbol{M} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi & \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{bmatrix} \quad (6.14)$$

[4] Computer library functions such as asin() return the principal value only. The second solution for θ may be obtained by subtracting the principal value from π.

6.7.9 From matrix to Tait-Bryan angle *z-y-x* convention

Given a rotation matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may be solved for the principal rotation factors $\boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle$, and therefore solved for angles (ψ, θ, φ). The solution is given in Table 6.7.

Table 6.7 — Principal rotation factors for the Tait-Bryan angle *z-y-x* convention

Case	Principal rotation factors for $\boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle$ (all angles modulo 2π, $M = [a_{ij}]$)		
$a_{31} \neq \pm 1$	$\theta = \arcsin(-a_{31})$ [principal value] $-\pi/2 < \theta < \pi/2$	$\varphi = \operatorname{arctan2}(a_{32},\ a_{33})$	$\psi = \operatorname{arctan2}(a_{21},\ a_{11})$
	$\theta = \arcsin(-a_{31})$ [π – principal value] $\pi/2 < \theta < 3\pi/2$	$\varphi = \operatorname{arctan2}(-a_{32},\ -a_{33})$	$\psi = \operatorname{arctan2}(-a_{21},\ -a_{11})$
$a_{31} = -1$	$\theta = \pi/2$	$\varphi = \operatorname{arctan2}(a_{12},\ a_{13}) + \psi$	any value of ψ
$a_{31} = +1$	$\theta = -\pi/2$	$\varphi = \operatorname{arctan2}(-a_{12},\ -a_{13}) - \psi$	any value of ψ

The cases $a_{31} = -1$ with $\theta = \frac{\pi}{2 \text{ and}} a_{31} = +1$ with $\theta = -\pi/2$ are gimbal lock cases (see Equation 6.9). The corresponding rotations and simplified matrices are:

Case $a_{31} = -1, \theta = -\pi/2$: $\boldsymbol{R}_z\langle\psi - \varphi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle$ $$M = \begin{bmatrix} 0 & \sin(\psi-\varphi) & \cos(\psi-\varphi) \\ 0 & \cos(\psi-\varphi) & -\sin(\psi-\varphi) \\ -1 & 0 & 0 \end{bmatrix}$$

Case $a_{31} = +1, \theta = +\pi/2$: $\boldsymbol{R}_z\langle\psi + \varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle$ $$M = \begin{bmatrix} 0 & \sin(\psi+\varphi) & \cos(\psi+\varphi) \\ 0 & \cos(\psi+\varphi) & -\sin(\psi+\varphi) \\ -1 & 0 & 0 \end{bmatrix}$$

For this reason, in the case of $a_{31} = -1$ only the difference of the other two angles is determined as $\varphi - \psi = \operatorname{arctan2}(a_{12},\ a_{13})$. Therefore, all values are valid for ψ if we set $\varphi = \operatorname{arctan2}(a_{12},\ a_{13}) + \psi$. The case $a_{31} = +1$ is similar to the previous case with the sum of the angles determined by $\varphi + \psi = \operatorname{arctan2}(-a_{12},\ -a_{13})$.

As seen in the preceding tables, the three-angle sequence corresponding to a given rotation or orientation operator is not unique modulo 2π. Two sequences, $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ of ***x-y-z*** principal factors specify the same operator if they satisfy one the criteria specified in Table 6.8.

Table 6.8 — Equivalence of *x-y-z* or *z-y-x* principal rotation factor sequences

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ for principal factor *z-y-x* or *x-y-z* sequences**
$\theta_1 = \theta_2$	$\varphi_1 = \varphi_2,\ \ \psi_1 = \psi_2 [\theta_1 \neq \pm\pi/2 \neq \theta_2]$ (in)equalities π

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences** $(\varphi_1, \theta_1, \psi_1)$ **and** $(\varphi_2, \theta_2, \psi_2)$ **for principal factor** ***z-y-x*** **or** ***x-y-z*** **sequences**
$\lvert\theta_1 + \theta_2\rvert = \pi$	$\lvert\varphi_2 - \varphi_1\rvert = \pi, \quad \lvert\psi_2 - \psi_1\rvert = \pi[\theta_1 \neq \pm\pi/2 \neq \theta_2]$ (in)equalities modulo 2π
$\theta_1 = \theta_2 = \frac{\pi}{2}$	$\varphi_1 + \psi_1 = \varphi_2 + \psi_2$ ***x-y-z*** case, equality modulo 2π $\varphi_1 - \psi_1 = \varphi_2 - \psi_2$ ***z-y-x*** case, equality modulo 2π
$\theta_1 = \theta_2 = -\frac{\pi}{2}$	$\varphi_1 - \psi_1 = \varphi_2 - \psi_2$ ***x-y-z*** case, equality modulo 2π $\varphi_1 + \psi_1 = \varphi_2 + \psi_2$ ***z-y-x*** case, equality modulo 2π

6.7.10 From axis-angle to quaternion

Given a rotation in axis-angle form (n_1, n_2, n_3, θ), the corresponding unit quaternion is:

$$\boldsymbol{q} = (\cos(\theta/2)\,,\ \sin(\theta/2)\,\boldsymbol{n})$$

where:

$$\boldsymbol{n} = (n_1, n_2, n_3).$$

6.7.11 From quaternion to axis-angle

Given a unit quaternion in scalar vector form $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = [e_1,\ e_2,\ e_3]^{\mathrm{T}}$, the corresponding axis-angle representation is computed as in Equation 6.10.

$\boldsymbol{n} = \frac{1}{\|\boldsymbol{e}\|}\boldsymbol{e}$ is a unit vector in 3D space.

$$\theta = 2 \cdot \operatorname{arctan2}\left(\sqrt{e_1^2 + e_2^2 + e_3^2},\ e_0\right).$$

6.7.12 From quaternion to matrix

Given a unit quaternion in scalar vector form $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = [e_1, e_2, e_3]^T$, the corresponding matrix representation is:

$$\boldsymbol{M} = \begin{bmatrix} 1 - 2(e_2^2 + e_3^2) & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & 1 - 2(e_1^2 + e_3^2) & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & 1 - 2(e_1^2 + e_2^2) \end{bmatrix} \tag{6.15}$$

6.7.13 From matrix to quaternion

Given a rotation matrix $\boldsymbol{M}$ with elements a_{ij}, the corresponding quaternion $\boldsymbol{q}$ is computed as follows:

$$e_0^2 = \frac{1}{4}(1 + a_{11} + a_{22} + a_{33})$$

if $e_0^2 > 0$,

$$\boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{4e_0} \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix},$$

else

$$e_0 = 0, \quad e_1^2 = -\frac{1}{2}(a_{22} + a_{33}),$$

if $e_1^2 > 0$,

$$e_2 = \frac{a_{12}}{2e_1}, \quad e_3 = \frac{a_{13}}{2e_1},$$

else

$$e_1 = 0, \quad e_2^2 = \frac{1}{2}(1 - a_{33}),$$

if $e_2^2 > 0$,

$$e_3 = \frac{a_{23}}{2e_2}$$

else

$$e_2 = 0, \quad e_3 = 1.$$

A rotationally equivalent quaternion is $-\boldsymbol{q}$.

6.7.14 From Euler angle *z-x-z* convention to quaternion

Given the Euler angle *z-x-z* convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_z}\langle\alpha\rangle \circ \boldsymbol{R_x}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$, the corresponding quaternion is:

$$\boldsymbol{q} = (\cos(\alpha/2)\,,\ \sin(\alpha/2)\,\boldsymbol{z})\ (\cos(\beta/2)\,,\ \sin(\beta/2)\,\boldsymbol{x})\ (\cos(\gamma/2)\,,\ \sin(\gamma/2)\,\boldsymbol{z}).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0,\ \boldsymbol{e})$$

where:

$$e_0 = \cos((\alpha + \gamma)/2)\cos(\beta/2),$$

$$\boldsymbol{e} = (\cos((\alpha - \gamma)/2)\sin(\beta/2)\,,\quad \sin((\alpha - \gamma)/2)\sin(\beta/2)\,,\quad \sin((\alpha + \gamma)/2)\cos(\beta/2))$$

6.7.15 From quaternion to Euler angle *z-x-z* convention

Given a unit quaternion in scalar vector form $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = [e_1, e_2, e_3]^T$, the corresponding Euler angle *z–x–z* convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_z}\langle\alpha\rangle \circ \boldsymbol{R_x}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$ are computed as follows:

if $0 < (e_1^2 + e_2^2) < 1$:

$$\alpha = \operatorname{arctan2}\big((e_1e_3 + e_0e_2),\ -(e_2e_3 - e_0e_1)\big)$$

$$\beta = \arccos\big(1 - 2(e_1^2 + e_2^2)\big) \quad \text{principal value: } 0 < \beta < \pi$$

$$\gamma = \operatorname{arctan2}\big((e_1e_3 - e_0e_2),\ (e_2e_3 + e_0e_1)\big)$$

if $(e_1^2 + e_2^2) = 0$:

$$\beta = 0 \text{ and } \alpha + \gamma = \operatorname{arctan2}\left((e_1e_2 - e_0e_3),\ \tfrac{1}{2} - (e_2^2 + e_3^2)\right).$$

if $(e_1^2 + e_2^2) = 1$:

$$\beta = \pi \text{ and } \alpha - \gamma = \operatorname{arctan2}\left((e_1 e_2 - e_0 e_3),\ \tfrac{1}{2} - (e_2^2 + e_3^2)\right).$$

The solution in the first case is not unique, see Table 6.5. The last two cases are Euler angle gimbal lock cases.

6.7.16 From Tait-Bryan angle *x-y-z* convention to quaternion

Given the Tait-Bryan angle *x-y-z* convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_x}\langle\varphi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_z}\langle\psi\rangle$, the corresponding quaternion is:

$$\boldsymbol{q} = (\cos(\varphi/2),\ \sin(\varphi/2)\,\boldsymbol{x})\ (\cos(\theta/2),\ \sin(\theta/2)\,\boldsymbol{y})\ (\cos(\psi/2),\ \sin(\psi/2)\,\boldsymbol{z}).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$$

where:

$$e_0 = \cos(\varphi/2)\cos(\theta/2)\cos(\psi/2) - \sin(\varphi/2)\sin(\theta/2)\sin(\psi/2)$$

$$e_1 = \cos(\varphi/2)\sin(\theta/2)\sin(\psi/2) + \sin(\varphi/2)\cos(\theta/2)\cos(\psi/2)$$

$$e_2 = \cos(\varphi/2)\sin(\theta/2)\cos(\psi/2) - \sin(\varphi/2)\cos(\theta/2)\sin(\psi/2)$$

$$e_3 = \cos(\varphi/2)\cos(\theta/2)\sin(\psi/2) + \sin(\varphi/2)\sin(\theta/2)\cos(\psi/2)$$

6.7.17 From quaternion to Tait-Bryan angle *x-y-z* convention

Given a unit quaternion in scalar vector form $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = [e_1, e_2, e_3]^T$, the corresponding Tait-Bryan angle *x-y-z* convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_x}\langle\varphi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_z}\langle\psi\rangle$ are computed as follows:

If $2(e_1 e_3 + e_0 e_2) \neq \pm 1$:

$$\varphi = \operatorname{arctan2}\left((e_2 e_3 - e_0 e_1),\ \frac{1}{2} - (e_1^2 + e_2^2)\right)$$

$$\theta = \arcsin\big(2(e_1 e_3 + e_0 e_2)\big) \quad \text{principal value: } -\pi/2 < \theta < \pi/2$$

$$\psi = \operatorname{arctan2}\left(-(e_1 e_2 - e_0 e_3),\ \frac{1}{2} - (e_2^2 + e_3^2)\right)$$

If $2(e_1 e_3 + e_0 e_2) = +1$:

$$\theta = -\pi/2 \text{ and } \varphi + \psi = \operatorname{arctan2}\big((e_1 e_2 + e_0 e_3),\ -(e_1 e_3 - e_0 e_2)\big).$$

If $2(e_1 e_3 + e_0 e_2) = -1$:

$$\theta = \pi/2 \text{ and } \varphi - \psi = \operatorname{arctan2}\big((e_1 e_2 + e_0 e_3),\ (e_1 e_3 - e_0 e_2)\big).$$

The solution in the first case is not unique, see Table 6.8. The last two cases are Euler angle gimbal lock cases.

6.7.18 From Tait-Bryan angle *z-y-x* convention to quaternion

Given the Tait-Bryan angle $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_z}\langle\psi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_x}\langle\varphi\rangle$, the corresponding quaternion is:

$$\boldsymbol{q} = (\cos(\psi/2),\ \sin(\psi/2)\,\boldsymbol{z})\ (\cos(\theta/2),\ \sin(\theta/2)\,\boldsymbol{y})\ (\cos(\varphi/2),\ \sin(\varphi/2)\,\boldsymbol{x}).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$$

where:

$$e_0 = \cos(\psi/2)\cos(\theta/2)\cos(\varphi/2) + \sin(\psi/2)\sin(\theta/2)\sin(\varphi/2)$$

$$e_1 = \cos(\psi/2)\cos(\theta/2)\sin(\varphi/2) - \sin(\psi/2)\sin(\theta/2)\cos(\varphi/2)$$

$$e_2 = \cos(\psi/2)\sin(\theta/2)\cos(\varphi/2) + \sin(\psi/2)\cos(\theta/2)\sin(\varphi/2)$$

$$e_3 = \sin(\psi/2)\cos(\theta/2)\cos(\varphi/2) - \cos(\psi/2)\sin(\theta/2)\sin(\varphi/2)$$

6.7.19 From quaternion to Tait-Bryan angle *z-y-x* convention

Given a unit quaternion in scalar vector form $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = [e_1, e_2, e_3]^T$, the corresponding Tait-Bryan angle $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention space-fixed equivalent of body-fixed consecutive rotations $\boldsymbol{R_z}\langle\psi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_x}\langle\phi\rangle$ are computed as follows:

If $2(e_1e_3 - e_0e_2) \neq \pm 1$:

$$\varphi = \operatorname{arctan2}\big((e_2e_3 + e_0e_1),\ \ 1/2 - (e_1^2 + e_2^2)\big)$$

$$\theta = \arcsin\big(-2(e_1e_3 - e_0e_2)\big) \text{ principal value: } -\pi/2 < \theta < \pi/2$$

$$\psi = \operatorname{arctan2}\big((e_1e_2 + e_0e_3),\ \ 1/2 - (e_2^2 + e_3^2)\big)$$

If $2(e_1e_3 - e_0e_2) = +1$:

$$\theta = -\pi/2 \text{ and } \varphi + \psi = \operatorname{arctan2}\big((e_1e_2 - e_0e_3),\ (e_1e_3 + e_0e_2)\big).$$

If $2(e_1e_3 - e_0e_2) = -1$:

$$\theta = \pi/2 \text{ and } \varphi - \psi = \operatorname{arctan2}\big((e_1e_2 - e_0e_3),\ (e_1e_3 + e_0e_2)\big).$$

The solution in the first case is not unique, see Table 6.8. The last two cases are Euler angle gimbal lock cases.