The following essay is prepared for discussion of the ideas of scale and origin in map projections for the committee writing the document, Spatial Reference Model (ISO/IEC CD 18026)

Let (u, v) be the components of the point in coordinate-space that corresponds to (x, y, z) in position-space. The correspondence defines a surface parametrically:

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \\ z &= z(u, v) \end{aligned}$$

The first fundamental form of the surface defines the square of differential arclength for the surface and is given by

$$E\,\mathrm{du}^2 + 2\,F\,\mathrm{du}\,\mathrm{dv} + G\,\mathrm{dv}^2$$

where

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ F &= \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) + \left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial z}{\partial v}\right) \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned}$$

The first fundamental form of the (u, v) plane defines the square of differential arclength in the (u, v) plane and is given by

$$\mathrm{du}^2 + \mathrm{dv}^2$$

The <u>point scale</u> at the point (u_0, v_0) in the direction $(\mathrm{du}, \mathrm{dv})$ is the ratio of differential distance on the (u, v) plane to differential arclength on the surface:

$$\text{point scale} = \sqrt{\frac{\mathrm{du}^2+\mathrm{dv}^2}{E\,\mathrm{du}^2+2\,F\,\mathrm{du}\,\mathrm{dv}+G\,\mathrm{dv}^2}}$$

where E, F, and G are evaluated at (u_0, v_0).

If the mapping $(u, v) \to (x, y, z)$ is conformal, then $E = G$ and $F = 0$ and the point scale equals $\frac{1}{\sqrt{E}}$ and is independent of the direction $(\mathrm{du}, \mathrm{dv})$.

Although the point scale varies by location and direction (generally) throughout the map, it is customary to pick a particular value of the point scale, perhaps a round number amidst a large or small variation in point scales, perhaps the point scale at a special point in a special direction, and call it the <u>map scale</u>. However decided, the map scale must be in the range of the function that computes point scale. Then, a useful quantity called the <u>scale factor</u> can be defined:

$$\text{scale factor} = \frac{\text{point scale}}{\text{map scale}}$$

If the mapping is not conformal, the scale factor will depend on direction as well as location.

Other terms that may be used for point scale: local scale.

Other terms that may be used for map scale: nominal scale.

Other terms that may be used for scale factor: magnification, point magnification, scale change factor

For conformal maps, curves on the surface on which the point scale is constant are called lines of constant scale. And, curves on the surface on which the point scale equals the map scale are called lines of true scale. Similar definitions are made for non-conformal maps with the additional requirement that the direction for the calculation of the point scale is tangent to the curve.

It may happen that the lines of constant scale are found among the coordinate curves, i.e. the lines $u = \text{constant}$ and the lines $v = \text{constant}$. If the curve $u = u_0$ is a line of true scale, then the value u_0 may be called a (u-value) of true scale. Likewise, if the curve $v = v_0$ is a line of true scale, then the value v_0 may be called a (v-value) of true scale.

If the surface is an ellipsoid model of the Earth, and the coordinates (u, v) are related to the geodetic coordinates (λ, ϕ) by mapping equations, added convenience may arise from the situation that the lines of constant scale are found among the meridians and parallels, i.e. the lines $\lambda = \text{constant}$ and the lines $\phi = \text{constant}$. If the meridian $\lambda = \lambda_0$ is a line of true scale, then the value λ_0 may be called a longitude of true scale. Likewise, if the parallel circle $\phi = \phi_0$ is a line of true scale, then the value ϕ_0 may be called a latitude of true scale. For example, for the Mercator projection, there is a latitude of true scale. If it is not the equator, there are two latitudes of true scale, symmetric about the equator.

Longitude of true scale and latitude of true scale may or may not exist for a particular map projection. In any case, they should not be confused with the longitude of natural origin, and the latitude of natural origin which are the values of (λ, ϕ) respectively that correspond to the natural origin $(u, v) = (0, 0)$. At the discretion of the cartographer, the (u, v) coordinate system may be adjusted to avoid negative numbers for u and v in the area of interest. Let $u_F > 0$ and $v_F > 0$ be the offsets chosen for u and v respectively that will accomplish this. Then, the values of (λ, ϕ) that correspond to the false origin $(u, v) = (u_F, v_F)$ are called the longitude of false origin, and the latitude of false origin respectively.

The above discussion mixes cartographic choices with mathematical relationships. In practice, the cartographer may make decisions about these entities in an order different than the definitions are presented above. The settled form of the equations $(u, v) \to (x, y, z)$ may be the last issue decided.

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